

PERSONAL DATA

---

First Name: MICHELE  
Surname: RIMOLDI  
Born: CANTÙ (CO), 17/11/1984  
Nationality: ITALIAN  
Contact Address: DIPARTIMENTO DI SCIENZE MATEMATICHE  
Politecnico di Torino  
Corso Duca degli Abruzzi, 24  
10129 Torino, ITALY  
e-mail: WORK        michele.rimoldi@polito.it  
                 PERSONAL    michele.rimoldi@gmail.com  
Personal page: <http://michelerimoldi.altervista.org>

CURRENT POSITION

---

Since January 2017        Researcher (RTD-a)  
                                 Politecnico di Torino  
                                 Dipartimento di Scienze Matematiche "Giuseppe Luigi Lagrange"

PREVIOUS POSITIONS

---

Oct. 2016-Dec. 2016        "Emma e Giovanni Sansone" Junior visiting position  
                                 Centro di Ricerca Matematica "Ennio De Giorgi"  
                                 Scuola Normale Superiore (Pisa, Italy)  
Dec. 2015-Sept. 2016        Post-Doc fellowship (1 year)  
                                 of the Fondation Sciences Mathématiques de Paris (FSMP)  
                                 Laboratoire Analyse Géométrie et Applications (LAGA)  
                                 Institut Galilée  
                                 Université Paris 13 - Sorbonne Paris Cité  
Jan. 2014 - Nov. 2015        Assegno di Ricerca (2 years Post-Doc position),  
                                 Università degli Studi di Milano-Bicocca  
                                 Dipartimento di Matematica e Applicazioni  
Dec. 2011 - Dec. 2013        Assegno di Ricerca (2 years Post-Doc position),  
                                 under the supervision of Prof. Stefano Pigola,  
                                 Università degli Studi dell'Insubria (Como, Italy)  
                                 Dipartimento di Scienza e Alta Tecnologia

ACADEMIC HABILITATIONS

---

December 2014                National Italian habilitation as *Professore di II fascia*  
                                 in Geometry and Algebra. Sector: 01/A2.  
                                 Validity: 30/12/2014–30/12/2020

- February 2016                      Qualification Française aux fonctions de *Maître de conférences*.  
25ème section.
- March 2018                         National Italian habilitation as *Professore di II fascia*  
in Geometry and Algebra. Sector: 01/A2.  
Validity: 30/03/2018–30/03/2024

---

EDUCATION

- March 2012                         Ph.D. in Mathematics,  
Università degli Studi di Milano, Italy  
Thesis–title: *Rigidity results for*  
*Lichnerowicz Bakry–Émery Ricci tensors*  
Supervisor: Prof. Stefano Pigola
- September 2008                    Laurea specialistica (master degree) cum laude in Mathematics  
Università degli Studi dell’Insubria (Como, Italy)  
Thesis–title: *Caratterizzazione di modelli mediante*  
*equazioni differenziali*  
Supervisor: Prof. Stefano Pigola
- January 2007                        Laurea triennale (bachelor) cum laude in Mathematics  
Università degli Studi dell’Insubria (Como, Italy)  
Thesis–title: *Costruzione di superfici minime complete*  
*tra due piani paralleli in  $\mathbb{R}^3$*   
Supervisor: Prof. Stefano Pigola
- July 2003                             Diploma di Maturità  
Liceo Scientifico Galileo Galilei (Erba (CO), Italy)

---

PUBLICATIONS

- [MMR]    C. MANTEGAZZA, S. MONGODI, M. RIMOLDI  
*The Cotton tensor and the Ricci flow*  
GEOMETRIC FLOWS.  
**2** (2017), no. 1, 49-71, doi: 10.1515/geoff-2017-0001
- [IR2]     D. IMPERA, M. RIMOLDI  
*Rigidity results and topology at infinity of translating solitons*  
*of the mean curvature flow*  
COMMUNICATIONS IN CONTEMPORARY MATHEMATICS.  
**19**, 1750002 (2017), no. 6, [21 pages], doi: 10.1142/S021919971750002X
- [IR1]     D. IMPERA, M. RIMOLDI  
*Stability properties and topology at infinity of  $f$ -minimal hypersurfaces*  
GEOMETRIAE DEDICATA  
**178** (2015), no. 1, 21-47, doi: 10.1007/s10711-014-9999-6
- [RV1]    M. RIMOLDI, G. VERONELLI  
*Topology of steady and expanding gradient Ricci solitons via  $f$ -harmonic maps*  
DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS  
**31** (2013), no. 5, 623-638, doi: 10.1016/j.difgeo.2013.06.001.

- [PR2] S. PIGOLA, M. RIMOLDI  
*Complete self-shrinkers confined into some regions of the space*  
ANNALS OF GLOBAL ANALYSIS AND GEOMETRY  
**45** (2014), no. 1, 47-65, doi: 10.1007/s10455-013-9387-8
- [MR] P. MASTROLIA, M. RIMOLDI  
*Some triviality results for quasi-Einstein manifolds and Einstein warped products*  
GEOMETRIAE DEDICATA  
**169** (2014), no. 1, 225-237, doi: 10.1007/s10711-013-9852-3
- [R2] M. RIMOLDI  
*On a classification theorem for self-shrinkers*  
PROCEEDINGS OF THE AMERICAN MATHEMATICAL SOCIETY  
**142** (2014), no. 10, 3605-3613, doi: 10.1090/S0002-9939-2014-12074-0
- [MRR] P. MASTROLIA, M. RIGOLI, M. RIMOLDI  
*Some geometric analysis on generic Ricci solitons*  
COMMUNICATIONS IN CONTEMPORARY MATHEMATICS  
**15** (2013), no. 3, 1250058, 25 pp., doi: 10.1142/S0219199712500587.
- [CMMR] G. CATINO, C. MANTEGAZZA, L. MAZZIERI, M. RIMOLDI  
*Locally conformally flat quasi-Einstein manifolds*  
JOURNAL FÜR DIE REINE UND ANGEWANDTE MATHEMATIK  
**675** (2013), 181-189, doi: 10.1515/crelle.2011.183.
- [R1] M. RIMOLDI  
*A remark on Einstein warped products*  
PACIFIC JOURNAL OF MATHEMATICS  
**252** (2011), no. 1, 207-218, doi: 10.2140/pjm.2011.252.207
- [MRV] P. MASTROLIA, M. RIMOLDI, G. VERONELLI  
*Myers-type theorems and some related oscillation results*  
JOURNAL OF GEOMETRIC ANALYSIS  
**22** (2012), no. 3, 763-779, doi:10.1007/s12220-011-9213-0
- [PRRS] S. PIGOLA, M. RIGOLI, M. RIMOLDI, A. G. SETTI  
*Ricci almost solitons*  
ANNALI DELLA SCUOLA NORMALE SUPERIORE DI PISA. CLASSE DI SCIENZE  
**X** (2011), no. 4, 757-799, doi: 10.2422/2036-2145.2011.4.01
- [PRiS] S. PIGOLA, M. RIMOLDI, A. G. SETTI  
*Remarks on non-compact gradient Ricci solitons*  
MATHEMATISCHE ZEITSCHRIFT  
**268** (2011), no. 3-4, 777-790, doi: 10.1007/s00209-010-0695-4
- [PR1] S. PIGOLA, M. RIMOLDI  
*Characterizations of model manifolds by means of certain differential systems*  
CANADIAN MATHEMATICAL BULLETIN  
**55** (2012), 632-645, doi:10.4153/CMB-2011-134-0

## PREPRINTS

- [IRS] D. IMPERA, M. RIMOLDI, A. SAVO  
*Index and first Betti number of  $f$ -minimal hypersurfaces and self-shrinkers*  
(submitted, [arXiv:1803.08268](#))
- [IPR] D. IMPERA, S. PIGOLA, M. RIMOLDI  
*The Frankel property for self-shrinkers from the viewpoint of elliptic PDE's*  
(submitted, [arXiv:1803.02332](#))
- [RV2] M. RIMOLDI, G. VERONELLI  
*Extremals of Log Sobolev inequality on non-compact manifolds and Ricci soliton structures*  
(submitted, [arXiv:1605.09240v2](#))

## TALKS, SEMINARS AND POSTERS

- April 2017 UNIVERSITÀ DI TORINO, TORINO, ITALY  
*Differential Geometry Days*  
Invited talk:  
“Extremals of Log Sobolev inequalities on non-compact manifolds and Ricci soliton structures”
- February 2017 SCUOLA NORMALE SUPERIORE, PISA, ITALY  
*Workshop su varietà reali e complesse geometria, topologia e analisi armonica*  
Invited talk:  
“Extremals of Log Sobolev inequalities on non-compact manifolds and Ricci soliton structures”
- December 2016 SCUOLA NORMALE SUPERIORE, PISA, ITALY  
*Seminari di Analisi Geometrica*  
Seminar:  
“Extremals of Log Sobolev inequalities on non-compact manifolds and Ricci soliton structures”
- September 2016 UNIVERSITÀ DI CAGLIARI, CAGLIARI, ITALY  
*Complex and Riemannian Geometry days*  
Invited talk:  
“Extremals of Log Sobolev inequalities on non-compact manifolds and Ricci soliton structures”
- June 2016 UNIVERSITÉ PARIS 13, PARIS, FRANCE  
*Groupe de travail Analyse semi-classique*  
Seminar:  
“Stability properties and topology at infinity of  $f$ -minimal hypersurfaces”
- April 2016 UNIVERSITÉ PARIS 7, PARIS, FRANCE  
*Séminaire de Géométrie*  
Invited talk:  
“Properness and boundedness properties of complete self-shrinkers for the mean curvature flow”
- Mars 2016 UNIVERSITÉ PARIS 13, PARIS, FRANCE  
*Séminaire EDP non linéaires*

- Invited talk:  
*“Obstructions on the topology at infinity  
of translating solitons of the mean curvature flow”*
- April 2015 UNIVERSIDAD DE GRANADA, GRANADA, SPAIN  
*Workshop on Geometric Flows*  
Invited talk:  
*“Rigidity results and topology at infinity  
of translating solitons of the mean curvature flow”*
- February 2015 UNIVERSITÀ DI CAGLIARI, CAGLIARI, ITALY  
*Seminari di Matematica*  
Seminar:  
*“Complete self-shrinkers confined into some regions of the space”*
- November 2014 UNIVERSITÀ DEGLI STUDI DI MILANO, MILANO, ITALY  
*PDE’s and Global Analysis @ UniMi*  
Invited talk:  
*“Complete self-shrinkers confined into some regions of the space”*
- October 2014 LEVICO TERME, TRENTO, ITALY  
*Progressi Recenti in Geometria Reale e Complessa - IX*  
Invited talk:  
*“Complete self-shrinkers confined into some regions of the space”*
- February 2014 UNIVERSIDADE FEDERAL DO CEARÀ, FORTALEZA, BRASIL  
*VII Workshop on Geometric Analysis*  
Invited talk:  
*“Complete self-shrinkers confined into some regions of the space”*
- April 2011 CENTRO DI RICERCA MATEMATICA ENNIO DE GIORGI, PISA, ITALY  
*Ricci Solitons Days in Pisa 2011*  
Contributed talk:  
*“Triviality results for quasi-Einstein metrics  
and Einstein warped products”*
- February 2011 UNIVERSIDAD DE GRANADA, SPAIN  
*Spanish-Japanese Workshop on Differential Geometry in Granada*  
Poster:  
*“Triviality results for quasi-Einstein metrics  
and Einstein warped products”*
- January 2011 UNIVERSITÀ DEGLI STUDI DELL’INSUBRIA, ITALY  
*Seminari di Analisi Geometrica*  
Seminar:  
*“Triviality results for quasi-Einstein metrics  
and Einstein warped products”*
- January 2011 UNIVERSITÀ DEGLI STUDI DELL’INSUBRIA, ITALY  
*Seminari di Analisi Geometrica*  
Seminar:  
*“Quasi-Einstein metrics and Einstein warped products”*

## RESEARCH VISITS

---

February 2015	UNIVERSITÀ DI CAGLIARI, CAGLIARI, ITALY (1 WEEK)
February 2014	UNIVERSIDADE FEDERAL DO CEARÀ, FORTALEZA, BRASIL (2 WEEKS)
Sept.-Dec. 2012	INSTITUT HENRI POINCARÉ, PARIS, FRANCE (3 MONTHS) Guest for the research trimester in <i>Conformal and Kähler Geometry</i>
January 2011	SCUOLA NORMALE SUPERIORE, PISA, ITALY (1 WEEK)

## REFEREE ACTIVITY

---

Referee for: Proceedings of the Edinburgh Mathematical Society, Communications in Analysis and Geometry, Journal of Geometric Analysis, Journal of Mathematical Analysis and Applications, Illinois Journal of Mathematics, International Journal of Mathematics, Bulletin des Sciences Mathematiques, Annali di Matematica Pura ed Applicata, Differential Geometry and its Applications, Bulletin of the Belgian Mathematical Society, International Journal of Geometric Methods in Modern Physics, Mathematische Nachrichten, Publicationes Mathematicae Debrecen, Monatshefte für Mathematik, Beirträge zur Algebra und Geometrie.  
Reviewer for AMS Mathematical Reviews.

## PROFESSIONAL EXPERIENCE

---

2006-2007	Università degli Studi dell'Insubria Tutoring (16 hours) for the following courses: "Calcolo II", "Geometria I", "Algebra I"	COMO, ITALY
2011	Università degli Studi dell'Insubria Exercise classes (20 hours) for the course "Analisi Matematica II". Main lecturer: Prof. Alberto G. Setti	COMO, ITALY
2012	Università degli Studi dell'Insubria Exercise classes (10 hours) for the course "Geometria II". Main lecturer: Prof. Stefano Pigola	COMO, ITALY
2013	Università degli Studi dell'Insubria Exercise classes (16 hours) for the course "Geometria II". Main lecturer: Prof Stefano Pigola	COMO, ITALY
2015	Università degli Studi di Milano-Bicocca Exercise classes (24 hours) for the course "Geometria I". Main lecturer: Prof. Davide L. Ferrario	MILANO, ITALY
2017	Politecnico di Torino Exercise classes (20 hours) for the course "Algebra lineare e geometria". Main lecturer: Prof. C. Cumino	TORINO, ITALY
2017	Politecnico di Torino Exercise classes (20 hours) for the course "Algebra lineare e geometria". Main lecturer: Prof. M. Ferrarotti	TORINO, ITALY
2017	Politecnico di Torino Exercise classes (20 hours) for the course	TORINO, ITALY

	“Linear algebra and geometry”. Language: English. Main lecturer: Prof. E. Carlini	
2017	Politecnico di Torino Exercise classes (20 hours) for the course “Geometria differenziale e computazionale”. Main lecturer: Prof. E. Musso	TORINO, ITALY
2018	Politecnico di Torino Exercise classes (20 hours) for the course “Linear algebra and geometry”. Language: English. Main lecturer: Prof. E. Carlini	TORINO, ITALY

---

 FUNDING INFORMATION
 

---

2018	Recipient of the FFABR: ricercatori FONDO PER IL FINANZIAMENTO ATTIVITÀ BASE DI RICERCA FUNDED BY THE MIUR GRANT VALUE: 3.000 EU
2017-2020	Responsible for the STARTING GRANT PER GIOVANI RICERCATORI A TD A E B - I TRANCHE 2017 - RIMOLDI (53-RIL17RIMMIC) FUNDED BY THE POLITECNICO DI TORINO GRANT VALUE: 15.000 EU
2014-2015	Participant to the PRIN Project 2010-2011: VARIETÀ REALI E COMPLESSE: GEOMETRIA, TOPOLOGIA E ANALISI ARMONICA Coordinator: Prof. Fulvio Ricci
2014	Participant to the INdAM-GNAMPA Project : PROPRIETÀ ANALITICHE E SPETTRALI DI VARIETÀ PESATE E APPLICAZIONI Coordinator: Debora Impera
2013	Coordinator of the INdAM-GNAMPA Project: FUNZIONI E MAPPE ARMONICHE: MISURA QUANTITATIVA DELL'INSIEME CRITICO, REGOLARITÀ E PROBLEMI DI DIRICHLET GRANT VALUE: 2.500 EU
2010	Participant to the INdAM-GNAMPA Project: TEORIA DEL POTENZIALE NON LINEARE SU VARIETÀ RIEMANNIANE E APPLICAZIONI GEOMETRICHE Coordinator: Prof. Stefano Pigola

---

 SOCIETY MEMBERSHIPS
 

---

Since 2010	Member of INdAM-GNAMPA
------------	------------------------

## OTHER ACTIVITIES

Member of the organizing committee of the conference:  
 GEOMETRIC ANALYSIS, SUBMANIFOLDS  
 AND GEOMETRY OF PDE'S  
 To be held at Politecnico di Torino, Italy, September, 2019  
 Scientific Committee: C. Arezzo, G. Besson, G. Manno, E. Musso,  
 S. Salamon, C. Terng.

Member of the organizing committee of the mini-school/workshop:  
 PLURIPOTENTIAL THEORY, GEOMETRIC ANALYSIS  
 AND CALIBRATED GEOMETRY  
 To be held at Università di Torino, Italy, May 2-4, 2018

Member of the organizing committee of the school:  
 GEOMETRIC ANALYSIS ON RIEMANNIAN AND SINGULAR  
 METRIC MEASURE SPACES  
 Como, Italy, July 11-15, 2016  
 Mini-courses by: H.-D. Cao, N. Gigli, T. Ilmanen, R. Schoen, C. Sormani,  
 X.-P. Zhu.  
 Scientific Committee: G. Besson, S. Pigola, A. Setti, M. Troyanov.

Member of the organizing committee of the workshop:  
 A GEOMETRY DAY IN COMO 2015  
 Como, Italy, Jan 9, 2015  
 Talks by: Z. Djadli, A. Malchiodi, L. Mari, B. Nelli,  
 R. Paoletti.

Member of the organizing committee of the workshop:  
 A GEOMETRY DAY IN COMO 2014  
 Como, Italy, Jan 10, 2014  
 Talks by: F. Bonsante, F. Coda-Marques, I. Holopainen, N. Gigli,  
 C. Mantegazza, M. Rigoli.

Member of the organizing committee of the school:  
 GEOMETRIC ANALYSIS ON RIEMANNIAN AND SINGULAR METRIC SPACES  
 Como, Italy, Sept 30-Oct 5, 2013  
 Mini-courses by: S. Alexander, G. Carron, E. Hebey, U. Lang, A. Neves  
 Scientific Committee: G. Besson, S. Pigola, A. Setti, M. Troyanov.

## CONFERENCES, WORKSHOPS AND SUMMER SCHOOLS

February 2018	SCUOLA NORMALE SUPERIORE, PISA, ITALY <i>Workshop su varietà reali e complesse: geometria, topologia e analisi armonica</i>
April 2017	UNIVERSITÀ DI TORINO, TORINO, ITALY <i>Differential Geometry Days</i>
February 2017	SCUOLA NORMALE SUPERIORE, PISA, ITALY <i>Workshop su varietà reali e complesse: geometria, topologia e analisi armonica</i>



---

September 2016	UNIVERSITÀ DI CAGLIARI, CAGLIARI, ITALY <i>Complex and Riemannian Geometry Days</i>
July 2016	LAKE COMO SCHOOL FOR ADVANCED STUDIES, COMO, ITALY <i>School in Geometric Analysis</i> <i>Geometric Analysis on Riemannian and singular metric measure spaces</i>
April 2015	UNIVERSIDAD DE GRANADA, GRANADA, SPAIN <i>Workshop on Geometric Flows</i>
March 2015	GOETHE UNIVERSITÄT, FRANKKUFURT AM MAIN, GERMANY <i>Flow(ers) &amp; Friends</i> <i>A workshop on Geometric Analysis</i>
January 2015	UNIVERSITÀ DEGLI STUDI DELL'INSUBRIA, ITALY <i>A geometry day in Como</i>
November 2014	UNIVERSITÀ DEGLI STUDI DI MILANO, MILANO, ITALY <i>PDE's and Global Analysis @ UniMi</i>
October 2014	LEVICO TERME (TN), ITALY <i>Progressi Recenti in Geometria Reale e Complessa - IX</i>
September 2014	VILLASIMIUS (CA), ITALY <i>New trends in Differential Geometry 2014</i>
June 2014	CENTRO DE GIORGI, PISA, ITALY <i>ERC School on Geometric Evolution Problems</i>
February 2014	UNIVERSIDADE FEDERAL DO CEARÀ, FORTALEZA, BRASIL <i>VII Workshop on Geometric Analysis</i>
January 2014	UNIVERSITÀ DEGLI STUDI DELL'INSUBRIA, ITALY <i>A geometry day in Como</i>
October 2013	INSTITUT FOURIER, GRENOBLE, FRANCE <i>Autumn days in Grenoble</i> <i>Curvature in metric spaces</i>
October 2013	LAKE COMO SCHOOL FOR ADVANCED STUDIES, COMO, ITALY <i>School in Geometric Analysis</i> <i>Geometric Analysis on Riemannian and singular metric spaces</i>
June 2013	UNIVERSIDAD DE GRANADA, SPAIN <i>Variational problems and Geometric PDE's</i>
Feb.-Mar. 2013	SCUOLA NORMALE SUPERIORE, PISA, ITALY <i>Workshop su varietà reali e complesse:</i> <i>geometria, topologia e analisi armonica</i>
January 2013	UNIVERSITÀ DEGLI STUDI DELL'INSUBRIA, ITALY <i>A geometry day in Como</i>
Sept.-Dec. 2012	INSTITUT HENRI POINCARÉ, PARIS, FRANCE <i>Research trimester in Conformal and Kähler geometry</i>

June 2012	ICTP, TRIESTE, ITALY <i>ICTP-ESF School and Conference on Geometric Analysis</i>
September 2011	UNIVERSIDAD DE GRANADA, SPAIN <i>VI International Meeting on Lorentzian Geometry</i>
May 2011	UNIVERSITÀ DI MILANO BICOCCA, ITALY <i>Geometria in Bicocca</i>
April 2011	CENTRO DE GIORGI, PISA, ITALY <i>Ricci Solitons Days in Pisa 2011</i>
February 2011	UNIVERSITÀ DI MILANO BICOCCA, ITALY 1st Bicocca HART <i>Harmonic Analysis and Related Topics</i>
February 2011	UNIVERSIDAD DE GRANADA, SPAIN <i>Spanish–Japanese Workshop on Differential Geometry</i>
September 2010	VERBANIA, ITALY Riemann International School of Mathematics <i>Nonlinear Differential Equations</i>
June–July 2010	CETRARO (CS), ITALY CIME course on <i>Ricci Flow and Geometric Applications</i>
May 2010	DRESDEN UNIVERSITY OF TECHNOLOGY, GERMANY The 8th AIMS Conference on <i>Dynamical Systems, Differential Equations and Applications</i>
September 2009	UNIVERSITÀ DI CAGLIARI, ITALY <i>A harmonic map fest</i>
June 2009	CENTRO DE GIORGI, PISA, ITALY Research Trimester on <i>Geometric flows and Geometric Operators</i>
April 2009	VERBANIA, ITALY Riemann International School of Mathematics <i>Advances in number theory and geometry</i>

## RESEARCH ACTIVITY

So far, my research activity has mainly dealt with Riemannian Geometry, Geometric Analysis, and Geometry of PDE's. In particular, I've been focusing on the following topics:

- **Ricci solitons.**([PRiS], [PRRS], [MRR], [RV1], [RV2]) Let  $(M, \langle \cdot, \cdot \rangle)$  be a Riemannian manifold. A Ricci soliton structure on  $M$  is the choice of a smooth vector field  $X$  (if any) satisfying the soliton equation  $Ric + \frac{1}{2}\mathcal{L}_X \langle \cdot, \cdot \rangle = \lambda \langle \cdot, \cdot \rangle$ . In the special case where  $X = \nabla f$  for some smooth function  $f : M \rightarrow \mathbb{R}$  we say that  $(M, \langle \cdot, \cdot \rangle, \nabla f)$  is a gradient Ricci soliton with potential  $f$ . In this case the soliton equation reads  $Ric_f = Ric + Hess(f) = \lambda \langle \cdot, \cdot \rangle$ .

Since the appearance of the seminal works of R. Hamilton, [18], and G. Perelman, [30], the study of gradient Ricci solitons has become the subject of a rapidly increasing investigation

mainly directed towards two goals, classification and triviality.

(i) In collaboration with Stefano Pigola and Alberto G. Setti, in [PRiS] we have studied triviality and rigidity results and curvature estimates for gradient Ricci solitons under weighted  $L^p$  conditions on the relevant quantities. These extend and generalize, often in a significant way, previous results (see e.g. [15, 31, 33, 42, 10]). Moreover, we show how techniques coming from stochastic analysis, such as stochastic completeness, in the form of the weak Omori–Yau maximum principle, parabolicity and  $L^p$ –Liouville type results, once translated into the setting of weighted geometry, are natural in the investigation of gradient Ricci solitons, and lead to elegant proofs of the above mentioned results.

(ii) In collaboration with Stefano Pigola, Marco Rigoli, and Alberto G. Setti, in [PRRS] we have introduced an extension of the concept of gradient Ricci soliton, the Ricci almost soliton, allowing  $\lambda$  in the soliton equation to be a generic smooth function on the weighted manifold  $M_f = (M, \langle \cdot, \cdot \rangle, e^{-f} d\text{vol})$ . In view of the fact that the soliton function  $\lambda$  is not necessarily constant, one expects that a certain flexibility on the almost soliton structure is allowed and, consequently, the existence of almost solitons is easier to prove than in the classical situation. This feeling is confirmed by a number of different examples of almost solitons. On the other hand we prove a rigidity result which indicates that almost solitons should reveal a reasonably broad generalization of the fruitful concept of classical soliton. In particular, one obtains that not every complete manifold supports an almost soliton structure. The investigation in [PRRS] is mainly concerned with triviality, pointwise curvature estimates and isolation phenomena for complete gradient Ricci almost solitons and the results follow from considering elliptic equations or inequalities for various geometric quantities on almost solitons and rely on analytical techniques. This is the same philosophy used e.g. by M. Eminent, G. La Nave and C. Mantegazza in [15] or in [PRiS]. More specifically, one sees that the differential (in)equalities at hand naturally involve the  $f$ –Laplacian. Since the almost soliton equation means precisely that the  $f$ –Bakry–Émery Ricci curvature  $Ric_f$  is proportional to the metric tensor, we are naturally led to introduce a number of weighted manifolds tools whose range of application go beyond the investigation of almost solitons. An important instance of these tools is represented by the Omori–Yau maximum principle for the  $f$ –Laplacian under both weighted Ricci lower bounds and weighted volume growth conditions. Moreover we extended to almost solitons also some topological results known in the classical case ([39, 18, 27, 15]).

(iii) While gradient Ricci solitons were widely studied, relatively little is known about generic Ricci solitons, that is, when the soliton field is not necessarily the gradient of a potential, and the majority of the results is concerned with the compact case. A first important difference is that, in the general case, we cannot make use of the weighted manifold structure which naturally arises when dealing with gradient solitons. The same applies for related concepts such as the Bakry–Émery Ricci tensor, whose boundedness from below with a suitable radial function, together with an additional assumption on the potential function gives rise to weighted volume estimates. These facts restrict the applicability of analytical tools such as the weak Omori–Yau maximum principle for the diffusion operator known as  $f$ –Laplacian, weighted  $L^p$ –Liouville–type theorems and a *a priori* estimate. Nevertheless, in the general case the soliton structure is encoded in the geometry of an appropriate operator  $\Delta_X$  that we shall call the  $X$ –Laplacian and that is defined on  $u \in C^2(M)$  by  $\Delta_X u = \Delta u - \langle X, \nabla u \rangle$ . Assuming a suitable growth condition on the vector field  $X$ , in collaboration with Paolo Mastroli and Marco Rigoli, in [MRR] we obtained scalar curvature and volume estimates and a gap theorem for the traceless Ricci tensor. These results permit to deduce some interesting geometric consequences.

(iv) In collaboration with Giona Veronelli, in [RV1], we studied  $f$ –harmonic maps from non-compact manifolds into non-positively curved ones. Notably, we proved existence and vanishing results which generalize to the weighted setting part of Schoen and Yau’s theory of harmonic maps, [34, 35]. As an application, we deduced information on the topology of manifolds with lower bounded Bakry–Émery Ricci tensor, and in particular of steady and expanding gradient Ricci solitons. Unlike most results so far known some topological theorem is obtained without requiring any assumption on the weight function  $f$ .

(v) In collaboration with Giona Veronelli, on [RV2], we established the existence of extremals for the Log Sobolev functional on complete non-compact manifolds with Ricci curvature bounded from below and strictly positive injectivity radius, under a condition near infinity. This generalizes a result previously obtained in [40] where it was assumed that the manifold had bounded geometry (i.e. the Riemann curvature tensor and all its covariant derivatives are bounded and that there is a uniform positive lower bound on the volume of geodesic balls of radius 1). When Ricci curvature is also bounded from above we get exponential decay at infinity of the extremals. As a consequence of these analytical results we establish, under the same assumptions, that non-trivial shrinking Ricci solitons support a gradient Ricci soliton structure. Our technique hence gives different conditions from those previously considered in [27] for the validity of a such a conclusion. On the way, we prove two results of independent interest: the existence of a distance-like function with uniformly controlled gradient and Hessian on complete non-compact manifolds with bounded Ricci curvature and strictly positive injectivity radius and a general growth estimate for the norm of the soliton vector field on manifolds with bounded Ricci curvature.

- **Quasi-Einstein manifolds.** ([R1], [MR], [CMMR]) Another possible generalization of Einstein manifolds and gradient Ricci solitons, first considered by J. Case, Y.-S. Shu and G. Wei, [8], are quasi-Einstein manifolds. These are weighted manifolds  $M_f$  such that their modified Bakry-Émery Ricci tensor  $Ric + Hess(f) - \frac{1}{k}df \otimes df$  is a constant multiple of the metric. The importance of these geometric structures comes from a problem (proposed in [2]) on the existence of Einstein manifolds realized as warped products. Indeed, in [26], D.-S. Kim and Y.-H. Kim proved a characterization of quasi-Einstein metrics as base metrics of Einstein warped product metrics.

(i) Exploiting this relationship, in [R1] and in [MR] (this latter written in collaboration with Paolo Mastrolia), we have obtained triviality and rigidity results for non-compact quasi-Einstein manifolds and Einstein warped products with non-compact bases and Einstein fibre. The proofs rely on the Omori-Yau maximum principle for the weighted Laplacian, a new gradient estimate for solutions of weighted Poisson-type equations and some Liouville-type theorems in the setting of weighted manifolds.

(ii) We have next faced the problem of the local characterization of locally conformally flat quasi-Einstein manifold. In particular, in collaboration with Giovanni Catino, Carlo Mantegazza, and Lorenzo Mazzieri, in [CMMR] we proved that any complete locally conformally flat quasi-Einstein manifold of dimension  $m \geq 3$  is locally a warped product with  $(m - 1)$ -dimensional fibers of constant curvature. Although the result was already known for gradient Ricci solitons (of which quasi-Einstein manifolds are a generalization), [41, 9, 6], the strategy of the proof is completely new and can be used as the main step to classify locally conformally flat shrinking and steady gradient Ricci solitons.

- **Self-shrinkers of the mean curvature flow.** ([R2], [PR2]) By a self-shrinker of the MCF we mean a connected, isometrically immersed hypersurface  $x : \Sigma^m \rightarrow \mathbb{R}^{m+1}$  whose mean curvature vector field  $\mathbf{H}$  satisfies the equation  $x^\perp = -\mathbf{H}$ . Self-shrinkers play an important role in the study of mean curvature flow, since they describe all possible blow ups at a given singularity of a MCF. Let  $f = \frac{|x|^2}{2}$  and denote by  $\mathbf{A}$  the second fundamental form of the immersion. In [R2] we observed that from the self-shrinker equation one gets that  $Ric_f \geq 1 - |\mathbf{A}|^2$ . This important estimate permits to apply, once again, the analytical tools developed so far in weighted geometry. Indeed the differential (in)equalities at hand on self-shrinkers involve the natural drifted Laplacian  $\Delta_f$  on  $\Sigma_f$ .

(i) The classification of mean convex self-shrinkers began with [24], where G. Huisken showed that the only smooth closed mean convex self-shrinkers are round spheres. In the complete non-compact case G. Huisken, [23], also proved a classification theorem saying that the only possible smooth mean convex embedded self-shrinkers with bounded  $|\mathbf{A}|$  and (extrinsic) polynomial volume growth are the cylindrical products  $C_{\sqrt{k}}^{k, m-k} = \mathbb{S}_{\sqrt{k}}^k \times \mathbb{R}^{m-k}$ ,  $k = 0, \dots, m$ . Recently T. Colding and W. Minicozzi, [13], showed that the hypothesis  $|\mathbf{A}|$  bounded can be dropped. The hypothesis of polynomial volume growth in [13] is used to show that various weighted integral converge in order to justify various integration by parts and it is

natural in the study of the singularities that a MCF goes through since any time-slice of a blow up of a closed MCF has polynomial growth. Nevertheless, looking only at the self-shrinker equation, it might be thought under what weaker conditions the conclusion in [13] still holds. In the main result in [R2] we replace polynomial volume growth with a weighted  $L^2$ -condition on  $|\mathbf{A}|$ . The viewpoint of weighted manifolds we adopt permits also to recover easily a classification result by H.D. Cao and H. Li, [6], where instead of the mean convexity condition it is considered a  $L^\infty$ -type condition on  $|\mathbf{A}|$ .

(ii) It was conjectured by H. D. Cao, [7], that every complete self-shrinker has extrinsic polynomial volume growth. By a result due to Q. Ding and Y. L. Xin, [14], later completed by X. Cheng and D. Zhou, [12], this is equivalent to the fact that the immersion is proper. Thus, by way of example, if Cao conjecture is true, then any complete self-shrinker in a ball of  $\mathbb{R}^{m+1}$  should be compact. In this direction, in collaboration with Stefano Pigola, in [PR2] we moved some steps in getting information on the geometry of non-compact bounded self-shrinkers (if any) obtaining natural and general restrictions that force these hypersurfaces to be compact (e.g. global boundedness conditions, pinching conditions at infinity and  $L^{p \geq m}$  conditions on  $|\mathbf{A}|$ ). More generally we tried to understand the geometry of self-shrinkers which are confined in a connected region bounded by some dilated cylinder  $\mathcal{C}_R^{k, m-k}$ . This was done proving a general strong maximum principle for self-shrinkers. Actually, when  $k = 0$ , it is reasonable that a complete self-shrinker has a certain homogeneous distribution around  $0 \in \mathbb{R}^{m+1}$  and, therefore, it should intersect every hyperplane through the origin. Moreover, by the maximum principle, the intersection should be transversal somewhere unless the self-shrinker is itself an hyperplane. For compact self-shrinkers these properties are easily verified and a similar conclusion can be obtained for complete self-shrinkers with a controlled extrinsic geometry. In [PR2] we pointed out natural geometric conditions that permit to recover the full conclusion of the compact case. Once we have understood that, to a certain extent, complete self-shrinkers intersect transversally a hyperplane through the origin, we were also able to deduce spectral information on the drifted Laplacian  $\Delta_f$  whenever the intersection is compact, and some (extrinsic) volume growth condition is satisfied.

- **$f$ -minimal hypersurfaces and translators of the MCF.** ([IR1], [IR2]) Following M. Gromov, [21], if we consider an isometrically immersed orientable hypersurface  $\Sigma^m$  in the weighted manifold  $M_f^{m+1}$ , we can define a generalization of the mean curvature vector field as  $\mathbf{H}_f = \mathbf{H} + (\nabla f)^\perp$ . Since the weighted structure on  $M$  induces a weighted structure on  $\Sigma$  we can consider the variational problem for the weighted area functional. From variational formulae, one can see that  $\Sigma$  is  $f$ -minimal, namely a critical point of the weighted area functional, if and only if  $\mathbf{H}_f$  vanishes identically. Clearly minimal hypersurfaces are a particular case of  $f$ -minimal hypersurfaces corresponding to the case  $f \equiv \text{const}$ . Moreover, self-shrinkers of the mean curvature flow are important examples of  $f$ -minimal hypersurfaces in the Euclidean space with the Gaussian density  $e^{-\frac{|x|^2}{2}}$ .

(i) The research on  $f$ -minimal hypersurfaces has just started and it has been already approached by many authors, see e.g. [17], [11], [16]. Much efforts have been devoted to the study of the stability properties. These are taken into account by spectral properties of the weighted Jacobi operator  $L_f = -\Delta_f - (|\mathbf{A}|^2 + \overline{\text{Ric}}_f(\nu, \nu))$ . The most up to date result, proved by X. Cheng, T. Mejia, and D. Zhou, [11], states that there exist no  $L_f$ -stable complete  $f$ -minimal hypersurfaces  $\Sigma$  immersed in a complete weighted Riemannian manifold  $M_f$  with  $\overline{\text{Ric}}_f \geq k > 0$ , provided  $\text{vol}_f(\Sigma) < +\infty$ . In collaboration with Debora Impera, in [IR1], we generalized this result considering growth conditions on the intrinsic weighted volume of geodesic balls. Furthermore, in the instability case, exploiting the oscillatory behaviour of solutions of some ODEs that naturally arise in this setting, we investigate general geometric restrictions for the finiteness of the weighted index of the  $f$ -minimal hypersurface, that is, the maximum dimension of the linear space of compactly supported deformations that decrease the weighted area up to second order. Moreover, exploiting a weighted version of a finiteness result and the adaptation to this setting of Li-Tam theory, we investigated the topology at infinity of  $f$ -minimal hypersurfaces. On the way, we prove a new comparison result in weighted geometry and we provided a general weighted  $L^1$ -Sobolev inequality for

hypersurfaces in Cartan–Hadamard weighted manifolds, satisfying suitable restrictions on the weight function and on its radial derivative.

(ii) By a translator of the MCF we mean a connected isometrically immersed complete hypersurface  $x : \Sigma^m \rightarrow \mathbb{R}^{m+1}$  whose mean curvature vector field  $\mathbf{H}$  satisfies the equation  $\mathbf{H} = v^\perp$  for some fixed unit length vector  $v \in \mathbb{R}^{m+1}$ . These hypersurfaces correspond to translating solitons of the MCF, and play a key role in the study of slowly forming singularities. Translators of the MCF turn out to be  $f$ -minimal hypersurfaces in the Euclidean space  $\mathbb{R}^{m+1}$  with the density  $e^{\langle v, x \rangle}$ . In collaboration with Debora Impera, in [IR2], we highlighted how the realm of weighted manifolds and  $f$ -minimal hypersurfaces can naturally give strong enough characterization and topological results for translators. About the firsts, we proved a rigidity theorem for  $f$ -stable translators under a weighted  $L^2$ -condition on the norm of the second fundamental form. This result, in particular, permits to strengthen previous rigidity results, see [36]. Concerning the latter, exploiting in an essential way the correspondence, discovered by K. Smoczyk, [37], between translators of MCF in  $\mathbb{R}^{m+1}$  and minimal hypersurfaces in the manifold  $\mathbb{R}^{m+1} \times \mathbb{R}$ , endowed with a suitable warped product metric, we obtain the validity of a weighted  $(m+1)$ -dimensional  $L^1$  Sobolev inequality on translators. The validity of this inequality permits to obtain very neat results on the topology at infinity of  $f$ -stable translators and translators with finite  $f$ -index. The main tools here are again weighted versions of the Li-Tam theory and of an abstract finiteness results, which were developed in [IR1]. Moreover, under the geometric assumption that the translator is contained in a halfspace determined by an hyperplane orthogonal to the translating direction  $v$ , we were able to guarantee the validity of the standard  $(m)$ -dimensional  $L^1$ -Sobolev inequality. This permit to get topological conclusion on translators also in case, instead of assuming the finiteness of the  $f$ -index, we are asking that the hypersurface has finite weighted total curvature.

- **Myers' type theorems.** ([MRV]) We analyzed the roots of the solutions of the special ODE  $u''(t) + k(t)u(t) = 0$ . The existence of (at least) a zero for a solution of this differential equation implies the compactness of the  $m$ -dimensional manifold  $M$ , once  $(m-1)k(t) = \text{Ric}_\gamma(t)$ , where  $\text{Ric}_\gamma$  is the radial Ricci curvature of  $M$ . Developing an idea of E. Calabi, [5], in collaboration with Paolo Mastrolia and Giona Veronelli, in [MRV] we gave a new compactness result for complete manifolds with Ricci curvature lower bounded by a negative constant, thus extending previous results in [1, 19, 28]. Similar techniques permit to analyze also the zeros of the more general problem  $(v(t)u'(t))' + k(t)v(t)u(t) = 0$ , for some “weight” function  $v(t)$ . Even in this case, we studied assumptions which guarantee existence of a zero and oscillation of the solutions. As previously observed by B. Bianchini, L. Mari and M. Rigoli, [3], an application of these results yields new estimates on the bottom of the spectrum and on the index of some Schrödinger operator on Riemannian manifolds.
- **Evolution by Ricci flow of geometric tensors.** ([MMR]) Complete gradient steady Ricci solitons play an important role in the study of Hamilton's Ricci flow as they correspond to translating solutions, and often arise as type II singularity models. Thus one is interested in classifying them and understanding their geometry. R. Bryant has discovered a steady Ricci soliton in dimension 3, which is rotationally symmetric and has positive curvature operator. Bryant's construction can be adapted to higher dimensions. Inspired by the uniqueness property in dimension 2 of the cigar soliton, in [31], Perelman conjectured that the Bryant soliton is (up to scaling) the only complete non-compact three-dimensional gradient steady Ricci soliton with positive sectional curvature which satisfies a non-collapsing assumption at infinity (namely  $\mathcal{K}$ -noncollapseness). In [6]. H.D. Cao and Q. Chen proved uniqueness in dimension  $m \geq 3$  under the additional assumption that  $(M^m, g)$  is locally conformally flat. The same result was proved independently by G. Catino and C. Mantegazza, [9], in dimension  $m \geq 4$ . Then, increasingly weaker conditions were considered, until, very recently, S. Brendle gave a proof of the conjecture without any further assumption, [4].

The proof strategy adopted in [9] to obtain uniqueness of locally conformally flat steady gradient non-flat Ricci solitons (for  $m \geq 4$ ) is based on the computation of the evolution of the Weyl tensor under the Ricci flow. Inspired by this technique, in collaboration with C.

Mantegazza and S. Mongodi, [CMM], we have computed the evolution of the Cotton and the Bach tensor under the Ricci flow of a Riemannian manifold, with particular attention to the three-dimensional case and we discussed some applications. Beyond their own interest, these computations suggest an alternative way to address Perelman's conjecture.

- **Obata's type characterization theorems.**([PR1]) We studied rigidity properties of Riemannian manifolds supporting solutions of particular second order differential systems. In particular, in collaboration with Stefano Pigola, in [PR1] we have considered more general differential systems than the classical ones of Obata's type ([29, 25, 38]), allowing the coefficient for the linear term to be a function (non-necessarily constant) of the distance function from a reference point. This permitted us to extend the characterization of space-forms to the case of model manifolds. We generalized also some metric rigidity results ([20]) by means of equations involving (gradient) vector fields.

### Bibliography for the research activity

- [1] W. Ambrose, *A theorem of Myers*, Duke Math. J. **24** (1957), 345–348.
- [2] A. Besse, *Einstein manifolds*, Springer-Verlag, Berlin-Heidelberg (1987).
- [3] B. Bianchini, L. Mari, M. Rigoli, *Spectral radius, index estimates for Schrödinger operators and geometric applications, (English summary)*, J. Funct. Anal. **256** (2009), no. 6, 1769–1820.
- [4] S. Brendle, *Rotationally symmetry of self-similar solutions to the Ricci flow* Invent. Math **194** (2013), no.3, 731–764.
- [5] E. Calabi, *On Ricci curvature and geodesics*, Duke Math. J. **34** (1967), 667-676.
- [6] H.-D. Cao, Q. Chen, *On locally conformally flat gradient steady Ricci solitons*, Trans. Amer. Math. Soc. **364** (2012), 2377-2391.
- [7] H.-D. Cao, H. Li, *A gap theorem for self-shrinkers of the mean curvature flow in arbitrary codimension*. Calc. Var. Partial Differential Equations. **46** (2013), 879–889.
- [8] J. Case, Y.-S. Shu, G. Wei, *Rigidity of quasi-Einstein metrics*, Differential Geom. Appl. **29** (2011), no. 1, 93–100.
- [9] G. Catino, C. Mantegazza, *The evolution of the Weyl tensor under the Ricci flow*. Ann. Inst. Fourier **61** (2011), no. 4, 1407–1435.
- [10] B.-L. Chen, *Strong uniqueness of the Ricci flow*, J. Differential Geom. **82** (2009), no. 2, 363–382.
- [11] X. Cheng, T. Mejia, and D. Zhou *Stability and compactness for complete  $f$ -minimal surfaces*. arXiv:1210.8076 (2012)
- [12] X. Cheng, D. Zhou, *Volume estimate about shrinkers*. Proc. Amer. Math. Soc. **141** (2013), 687-696.
- [13] T. Colding, W. Minicozzi, *Generic mean curvature flow I; generic singularities*. Ann. of Math. **175** (2012), 755-833.
- [14] Q. Ding, Y.L. Xin, *Volume growth, eigenvalue and compactness for self-shrinkers*. Asian J. Math. **17** (2012), 443-456
- [15] M. Eminent, G. La Nave, C. Mantegazza, *Ricci Solitons: the equation point of view*, Manuscripta Math. **127** (2008), 345–367.
- [16] J. M. Espinar *Manifolds with density, applications and gradient Schrödinger operators*. arXiv:1209.6162v6. (2012)
- [17] E. M. Fan, *Topology of three-manifolds with positive  $P$ -scalar curvature*. Proc. Amer. Math. Soc. **136** (2008), no. 9, 3255–3261.
- [18] M. Fernández-López, E. García-Río, *A remark on compact Ricci solitons*, Math. Ann. **340** (2008), 893–896.
- [19] G. J. Galloway, *Compactness criteria for Riemannian manifolds*, Proc. Amer. Math. Soc. **84** (1982), no. 1, 106–110.
- [20] E. García-Río, D. Kupeli and B. Unal, *On a differential equation characterizing Euclidean spheres*, J. Differential Equations **194** (2003), 287–299.

- [21] M. Gromov, *Isoperimetry of waists and concentration of maps*. Geom. Funct. Anal. **13** (2003), no. 1, 178–215.
- [22] R. S. Hamilton, *The Ricci flow on surfaces*, Mathematics and general relativity (Santa Cruz, CA, 1986), Contemp. Math., **71**. American Mathematical Society, Providence, 237–262 (1988).
- [23] G. Huisken, *Local and Global behaviour of hypersurfaces moving by mean curvature*, Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990), Proc. Sympos Pure Math. **54**, 175–191.
- [24] G. Huisken, *Asymptotic behaviour for singularities of the mean curvature flow*, J. Differential Geom. **31** (1990), no. 1, 285–299.
- [25] M. Kanai, *On a differential equation characterizing a Riemannian structure of a manifold*, Tokyo J. Math. **6** (1983), 143–151.
- [26] D.–S. Kim, Y.–H. Kim, *Compact Einstein warped product spaces with nonpositive scalar curvature*, Proc. Amer. Math. Soc. **131** (2003), no. 8, 2573–2576.
- [27] A. Naber, *Noncompact shrinking four solitons with nonnegative curvature*, J. Reine Angew. Math. **645** (2010), 125–153.
- [28] Z. Nehari, *Oscillation criteria for second-order linear differential equations*, Trans. Amer. Math. Soc. **85** (1957), 428–445.
- [29] M. Obata, *Certain conditions for a Riemannian manifold to be isometric with a sphere*, J. Math. Soc. Japan **14** (1962), 333–340.
- [30] G. Perelman, *Ricci flow with surgery on three manifolds*, arXiv:math.DG/0303109.
- [31] G. Perelman, *The entropy formula for the Ricci flow*, arXiv:math/0211159.
- [32] P. Petersen, W. Wylie. *Rigidity of gradient Ricci solitons*, Pacific J. Math. **241** (2009), no.2, 329–345.
- [33] P. Petersen, W. Wylie. *On the classification of gradient Ricci solitons*, Geom. Topol. **14** (2010), no. 4, 2277–2300.
- [34] R. Schoen, S.–T. Yau, *Harmonic Maps and the Topology of Stable Hypersurfaces and Manifolds with Non-negative Ricci Curvature*, Comm. Math. Helv. **51** (1976), 333–341.
- [35] R. Schoen, S.–T. Yau, *Compact group actions and the topology of manifolds with nonpositive curvature*, Topology **18** (1979), 361–380.
- [36] L. Shahriyari, *Translating graphs by mean curvature flow*, ProQuest LLC, Ann Arbor, MI, 2013, Thesis (Ph.D.)–The Johns Hopkins University.
- [37] K. Smoczyk, *A relation between mean curvature flow solitons and minimal submanifolds*, Math. Nachr. **229** (2001), 175–186.
- [38] Y. Tashiro, *Complete Riemannian manifolds and some vector fields*, Trans. Amer. Math. Soc. **117** (1965), 251–275.
- [39] W. Wylie, *Complete shrinking Ricci solitons have finite fundamental group*, Proc. Amer. Math Soc. **136** (2008), no. 5, 1803–1806.
- [40] Qi S. Zhang *Extremal of log Sobolev inequality and W-entropy on noncompact manifolds* J. Funct. Anal. **263** (2012) no.7 2051–2101.
- [41] Z.–H. Zhang, *Gradient shrinking solitons with vanishing Weyl tensor*, Pacific J. Math. **242** (2009), no. 1, 189–200.
- [42] Z.–H. Zhang, *On the completeness of gradient Ricci solitons*, Proc. Amer. Math. Soc. **137** (2009), no. 8, 2755–2759.

Torino, March 31, 2018

.....