Rigidity results for Lichnerowicz Bakry-Emery Ricci tensors

Ph.D. Thesis

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Our setting $\rightarrow$ Riemannian manifolds with density (weighted manifolds).

Let $(M, \langle\, , \rangle)$ be a complete m–dimensional Riemannian manifold. Fix an origin $o \in M$ and denote by

- $r(x)$ the distance function from o.
- B_r the geodesic ball of radius r centered at o.
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Given $f \in C^\infty(M)$ we can consider the **weighted manifold** $\left(M, \langle \, , \rangle, e^{-f} d\mathrm{vol}\right)$.
We set

$$\mathrm{vol}_f(B_r(p)) = \int_{B_r(p)} e^{-f} d\mathrm{vol}, \qquad \mathrm{vol}_f(\partial B_r(p)) = \int_{\partial B_r(p)} e^{-f} d\mathrm{vol}_{m-1}.$$

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We call f-**laplacian**, Δ_f, the diffusion operator defined on u by

$$\Delta_f u = e^f div\left(e^{-f} \nabla u\right) = \Delta u - \langle \nabla f, \nabla u\rangle$$

which is clearly symmetric on $L^2\left(M, \langle\, ,\rangle, e^{-f} d\mathrm{vol}\right)$.

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Note that if $k_1 \geq k_2$ then $\mathrm{Ric}_f^{k_1} \geq \mathrm{Ric}_f^{k_2}$ so that ,e.g.,

$$\mathrm{Ric}_f^k \geq \lambda \langle\, , \rangle \Rightarrow \mathrm{Ric}_f \geq \lambda \langle\, , \rangle\,.$$

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$$\frac{1}{2}\Delta_f|\nabla u|^2 \geq \frac{(\Delta_f u)^2}{m+k} + \langle \nabla u, \nabla \Delta_f u\rangle + \mathrm{Ric}_f^k(\nabla u, \nabla u).$$

In other words a Bochner formula holds for Ric_f^k, $k < \infty$ that looks like the Bochner formula for Ric of an $m + k$ dimensional manifold.

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- f–laplacian and weighted volume comparisons, (see [Setti '92], [Qian, '96], [Wei-Wylie, '09], [Mari-Rigoli-Setti, '10]).

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For example observe that

- If $(M, \langle\, ,\,\rangle, e^{-f}d\mathrm{vol})$ is complete and $\mathrm{Ric}_f^k \geq (m+k-1)c^2 > 0$, $k < \infty$. Then M is compact and $\mathrm{diam} \leq \frac{\pi}{c}$.
- If $(M, \langle\, ,\,\rangle, e^{-f}d\mathrm{vol})$ is complete and $\mathrm{Ric}_f^k \geq -(m+k-1)B^2$, $k < \infty$, for some constant $B > 0$. Then for every $r > 0$ there exists a constant such that

$$\mathrm{vol}_f(B_r(o)) \leq CB^{-(m+k-1)}\int_0^r (\sinh(Bt))^{(m+k-1)}dt.$$

The case $k = \infty$ is more critical and the difficulties that arise are absolutely non–technical.

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Example. The Gaussian space $(\mathbb{R}^m, \langle\, ,\, \rangle, e^{-f} d\mathrm{vol})$ with $f(x) = \frac{1}{2}A^2|x|^2$ for an arbitrary $A \in \mathbb{R}$, satisfies $\mathrm{Ric}_f = A^2 > 0$ but it is non–compact.

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$$\boxed{(M, \langle\, , \,\rangle, e^{-f} d\mathrm{vol}) \text{ complete, } \mathrm{Ric}_f \geq c^2 > 0 \quad \overset{[Wylie, '08]}{\Rightarrow} \quad |\pi_1(M)| < +\infty}$$

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$$\mathrm{vol}_f(B_r(0)) = C_{m-1} \int_0^r e^{\frac{A^2}{2}t^2} t^{m-1} dt \asymp e^{\frac{A^2}{2}t^2} t^{m-2}.$$

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Nevertheless there are again mutual relations beetween Ric_f bounds and vol_f–growth properties.
Let $(M, \langle\, , \rangle, e^{-f} d\mathrm{vol})$ be a complete weighted manifold, then

$$\boxed{\mathrm{Ric}_f \geq \lambda \in \mathbb{R} \quad \overset{[Qian,'96]}{\Rightarrow} \quad \mathrm{vol}_f(B_r(o)) \leq A + B \int_{R_0}^r e^{-\lambda t^2 + Ct} dt}$$

This can be extended to weighted volume estimates in case the radial lower bound is given by $D(1+r)^{-\mu}$, for some costants D, μ.
For example if $\mathrm{Ric}_f \geq D(1+r)^{-\mu}$, $D > 0$, $0 \leq \mu \leq 1$ then there exist costants $C_j > 0$ s.t. for every $r > 2$

$$\mathrm{vol}_f(\partial B_r(o)) \leq \begin{cases} C_1 e^{-C_2 r \log(1+r)} & \text{if } \mu = 1 \\ C_1 e^{-C_2 r^{2-\mu}} & \text{if } 0 \leq \mu < 1 \end{cases} \quad \text{and} \quad \mathrm{vol}_f(B_r(o)) \leq C_3$$

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In order to deal with the more general case

$$\mathrm{Ric}_f \geq -(m-1)\, G(r)$$

for a smooth positive function G on $\mathbb{R}_0^+$, even at the origin we have to impose some further condition on f or on its derivatives.

In particular let g be a solution on $\mathbb{R}_0^+$ of

$$\begin{cases} g'' - Gg \geq 0 \\ g(0) = 0, \quad g'(0) \geq 1. \end{cases}$$

Then in [Pigola, Rigoli, R., Setti, '10] we proved that there exists costants $B, C, D > 0$ s.t.

- $\begin{cases} \langle \nabla r, \nabla f \rangle \geq -\theta(r) \\ \theta \in C^0(\mathbb{R}_0^+), \ \theta' \geq 0 \end{cases} \quad \Rightarrow \quad vol_f\,(B_r(o)) \leq D \int_0^r g\,(t)^{m-1}\, e^{\int_0^t \theta(s)ds} dt.$
- $\begin{cases} \xi\,(r) \leq f \leq \omega\,(r) \\ \omega, \xi \in C^1\left(\mathbb{R}_0^+\right), \ \omega' \geq 0 \\ \xi'\,(r) \leq \omega'\,(r) \end{cases} \quad \Rightarrow \quad vol_f\,(B_r(o)) \leq C + B \int_{r_0}^r g\,(t)^{(m-1)+2((\omega-\xi)(t))}\, dt$

General idea: Analytic theorems for the Laplace–Beltrami operator on M, which are proved under metric–measure assumptions, using for example the divergence theorem, comparison arguments or heat semigroup methods, can be translitterated into theorems, under weighted metric–measure assumptions, for Δ_f on $(M, \langle\, , \,\rangle\, , e^{-f} d\mathrm{vol})$.

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- (Maximum principles) Let $\Omega \subseteq (M, \langle\, ,\, \rangle, e^{-f} d\mathrm{vol})$ be a connected domain. Then the following hold:
  (1) If $\Delta_f u \geq 0$ in Ω and $u(x_0) = \sup_\Omega u$ then $u \equiv u(x_0)$ in Ω.
  (2) If $\Delta_f u \leq cu$ in Ω, for a generic constant $c \in \mathbb{R}$, $u \geq 0$ in Ω and $u(x_0) = 0$ then $u \equiv 0$ in Ω.

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Given $(M, \langle\, , \rangle, e^{-f} d\mathrm{vol})$ we say that the **full Omori–Yau maximum principle for Δ_f** holds if given a C^2 function $u : M \to \mathbb{R}$ satisfying $\sup_M u = u^* < +\infty$, there exists a sequence $\{x_n\} \subset M$ along which

$$(i)\ u(x_n) \geq u^* - \frac{1}{n}, \quad (ii)\ |\nabla u(x_n)| \leq \frac{1}{n}, \quad \text{and} \quad (iii)\ \Delta_f u(x_n) \leq \frac{1}{n}.$$

On the other hand we say that the **weak Omori–Yau maximum principle for Δ_f** holds if for any u as above there exists a sequence $\{x_n\} \subset M$ along which (i) and (iii) hold.

Let $\left(M, \langle\, ,\, \rangle\, , e^{-f} d\mathrm{vol}\right)$ be a geodesically complete weighted manifold. From various results developed, among the others, by Pigola–Rigoli–Setti and Grigor'yan we have the following results.

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- Suppose the validity of the following weighted volume growth condition

$$\frac{r}{\log \mathrm{vol}_f\left(B_r\right)} \notin L^1\left(+\infty\right).$$

$$\Downarrow$$

1. The weak Omori–Yau maximum principle for Δ_f holds on M.
2. (**a–priori estimates**) Let $u \geq 0$ be a weak solution of

$$\Delta_f u \geq au + bu^{\sigma}$$

with constants $a \geq 0$, $b > 0$, $\sigma > 1$. Then

$$u \leq \left(\frac{a}{b}\right)^{\frac{1}{\sigma-1}}.$$

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- Suppose the validity of the following weighted area growth condition

$$\mathrm{vol}_f(\partial B_r)^{-1} \notin L^1\left(+\infty\right).$$

$$\Downarrow$$

1. The weighted manifold is f–**parabolic**, i.e., $\begin{cases} \Delta_f u \geq 0 \\ u^* = \sup_M u < +\infty \end{cases} \Rightarrow u \equiv \mathrm{const}.$

Combining these results with the comparison geometry we have discussed above we obtain that

- The weak Omori–Yau maximum principle for Δ_f and the a–priori estimate hold on a weighted manifold $(M, \langle\, ,\, \rangle, e^{-f} d\mathrm{vol})$ provided one of the following curvature assumptions is satisfied for some $\lambda \in \mathbb{R}$
  - (a) $\mathrm{Ric}_f^k \geq \lambda, \; k < \infty$;
  - (b) $\mathrm{Ric}_f \geq \lambda$.
- A weighted manifold $(M, \langle\, ,\, \rangle, e^{-f} d\mathrm{vol})$ is f–parabolic provided one of the following curvature assumptions is satisfied
  - (a) $Ric_f \geq \lambda > 0, \quad \lambda$ constant;
  - (b) $Ric_f \geq D(1+r)^{-\mu}$ with $D > 0$ and $0 \leq \mu \leq 1$.

- (L^p**–Liouville,** $1 < p < +\infty$) $\begin{cases} \Delta_f u \geq 0 \\ 0 \leq u \in L^p(M, e^{-f} d\mathrm{vol}) \end{cases} \Rightarrow u \equiv \mathrm{const.}$

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- (L^1**–Liouville**) $\begin{cases} \Delta_f u \geq 0 \\ 0 \leq u \in L^1(M, e^{-f} d\mathrm{vol}) \\ u(x) = O\left(e^{\alpha r(x)^{2-\epsilon}}\right) \qquad r(x) \to +\infty \end{cases} \Rightarrow u \equiv \mathrm{const.}$

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In general under weighted volume growth conditions, nothing can be said about the validity of the full–Omori Yau maximum principle for Δ_f. Nevertheless from a result in [Pigola–Rigoli–R.–Setti, '10] we obtain function–theoretic sufficient conditions for its validity. Let G be a smooth function on $[0, +\infty)$, even at the origin, satisfying

$$(i)\ G(0) > 0 \qquad (ii)\ G'(t) \geq 0 \text{ on } [0, +\infty)$$

$$(iii)\ G(t)^{-\frac{1}{2}} \notin L^1(+\infty) \qquad (iv)\ \limsup_{t \to +\infty} \frac{tG\left(t^{\frac{1}{2}}\right)}{G(t)} < +\infty.$$

- The full Omori–Yau maximum principle for Δ_f holds on a complete weighted manifold $(M, \langle\, ,\, \rangle, e^{-f} d\mathrm{vol})$ satisfying one of the following hypotheses
  - (a) $\mathrm{Ric}_f^k \geq -(m + k - 1)G(r)$;
  - (b) $\mathrm{Ric}_f \geq -(m-1)G(r)$ and $|\nabla f| \leq CG(r)^{1/2}$.

Given $(M, \langle\, ,\, \rangle)$ a Riemannian manifold, a **gradient Ricci soliton structure** on M is the choice of a smooth function f (if any) satisfying the soliton equation

$$\mathrm{Ric}_f = \lambda \langle\, ,\, \rangle\,.$$

for some constant $\lambda \in \mathbb{R}$. The gradient Ricci soliton $(M, \langle\, ,\, \rangle, \nabla f)$ is said to be shrinking, steady or expanding according to whether $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$.

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On the other hand, following the terminology in [Case–Shu–Wei, '08] we say that the Riemannian manifold $(M, \langle\, , \rangle)$ supports a *k*–**quasi–Einstein structure**, $k \in \mathbb{N}$, if there is some smooth function $f : M \to \mathbb{R}$ such that

$$\mathrm{Ric}_f^k = \lambda \langle\, , \rangle, \, k < \infty.$$

The importance of gradient Ricci solitons is due to Perelman's solution of Poincaré conjecture. They correspond to "self–similar" solutions to Hamilton's Ricci flow and often arise as limits of dilations of singularities developed along the Ricci flow.

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Example 1. The standard Euclidean space $(\mathbb{R}^m, \langle\, ,\, \rangle, \nabla f)$ with

$$f(x) = \frac{1}{2} A |x|^2 + \langle x, B \rangle + C,$$

for arbitrary $A \in \mathbb{R}$, $B \in \mathbb{R}^m$ and $C \in \mathbb{R}$. Note that f is the essentially unique solution of the equation $\mathrm{Hess}(f) = A \langle\, ,\, \rangle$ on $\mathbb{R}^m$. Infact by a "domain rigidity" result by Y. Tashiro also a kind of converse holds. See also [Pigola–R., '09].

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Example 2. The Riemannian product

$$\left(\mathbb{R}^m \times N^k, \langle\, , \,\rangle_{\mathbb{R}^m} + \langle\, , \,\rangle_{N^k}, \nabla f\right)$$

where $\left(N^k, \langle\, , \,\rangle_{N^k}\right)$ is any k–dimensional Einstein manifold with Ricci curvature $\lambda \neq 0$, and $f(t, x) : \mathbb{R}^m \times N^k \to \mathbb{R}$ is defined by

$$f(x, p) = \frac{\lambda}{2}|x|^2_{\mathbb{R}^m} + \langle x, B\rangle_{\mathbb{R}^m} + C, \tag{1}$$

with $C \in \mathbb{R}$ and $B \in \mathbb{R}^m$.

On the other hand the importance of k–quasi–Einstein manifolds comes from a problem (proposed by A. Besse) on the existence of Einstein manifolds realized as warped products. Indeed the following characterization holds.

Theorem (Kim–Kim, '03)

Let $M^m \times_u F^k$ be an Einstein warped product with Einstein constant λ, warping function $u = e^{-\frac{f}{k}}$ and Einstein fibre F^k. Then the weighted manifold $(M^m, g_M, e^{-f}d\mathrm{vol})$ satisfies the k–quasi–Einstein equation. Furthermore the Einstein constant μ of the fibre satisfies

$$\Delta f - |\nabla f|^2 = k\lambda - k\mu e^{\frac{2}{k}f}. \tag{2}$$

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On the other hand the importance of k–quasi–Einstein manifolds comes from a problem (proposed by A. Besse) on the existence of Einstein manifolds realized as warped products. Indeed the following characterization holds.

Theorem (Kim–Kim, '03)

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Table: Examples of k–quasi–Einstein manifolds

	$\lambda < 0$	$\lambda = 0$	$\lambda > 0$
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Actually in [Pigola–Rigoli–R.–Setti, '10] we also introduce an extension of the concept of gradient Ricci solitons, the **Ricci almost soliton**, allowing λ in the soliton equation to be a generic smooth function on the weighted manifold $(M, \langle\, , \,\rangle\, , e^{-f} d\mathrm{vol})$.

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Example. Let $g(t) : I \to \mathbb{R}^{+}$, $0 \in I \subseteq \mathbb{R}$, be the smooth function defined by

$$g(t) = g'(0)\mathrm{sn}_{-c}(t) + g(0)\mathrm{cn}_{-c}(t)$$

where

$$\mathrm{sn}_k(t) = \begin{cases} \frac{1}{\sqrt{-k}} \sinh\left(\sqrt{-k}t\right) & \text{if} \quad k < 0 \\ t & \text{if} \quad k = 0 \\ \frac{1}{\sqrt{k}} \sin\left(\sqrt{k}t\right) & \text{if} \quad k > 0 \end{cases} \qquad cn_k(t) = \dot{sn}_k(t).$$

Let $\left(\Sigma, (\, ,\,)_{\Sigma}\right)$ be an m–dimensional Einstein manifold satisfying

$${}^{\Sigma}\mathrm{Ric} = -(m-1)\left\{-g'(0)^2 + cg(0)^2\right\}(\, ,\,)_{\Sigma}\,.$$

Then, the warped product $M = I \times_g \Sigma$ is Einstein with ${}^{M}\mathrm{Ric} = -mc\langle\, , \rangle$ and it is an almost Ricci soliton with potential $f(t)$ and soliton function $\lambda(t)$ defined by

(3)
$$\begin{cases} f(t) = a\int_0^t g(s)\,ds + b \\ \lambda(t) = ag'(t) - mc, \end{cases}$$

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- Rigidity result $\Rightarrow$ basically there are no further examples of Einstein Ricci almost solitons.

Metric rigidity

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Let $(M, \langle\, , \rangle, \nabla f)$ be a complete gradient Ricci almost soliton with scalar curvature S and soliton function λ such that $\Delta\lambda \leq 0$ on M. Set $S_* = \inf_M S$, $\lambda_* = \inf_M \lambda$, $\lambda^* = \sup_M \lambda$.

(i) *If the almost soliton satisfies $-\infty < \lambda_* \leq \lambda \leq 0$, $\lambda \not\equiv 0$, then $m\lambda_* \leq S_* \leq 0$. Moreover, if $m \geq 3$ and there exists x_o such that $S(x_o) = S_* = m\lambda_*$, then the soliton is trivial and M is Einstein; while if $S(x_0) = S_* = 0$ for some $x_0 \in M$, then M is Ricci flat and isometric to $\mathbb{R}^m$.*

(ii) *If the almost soliton is a steady soliton then $S_* = 0$. Morever, if $m \geq 3$ and there exists x_0 such that $S(x_0) = 0$, then M is a cylinder over a totally geodesic hypersurface.*

(iii) *If the almost soliton satisfies $0 \leq \lambda \leq \lambda^*$, $\lambda \not\equiv 0$, then $0 \leq S_* \leq m\lambda^*$. Moreover if $m \geq 3$ and there exists x_o such that $S(x_o) = S_* = 0$ then M is isometric to $\mathbb{R}^m$. Finally if $S_* = m\lambda^*$ and $(M, \langle\, , \rangle, e^{-f}d\mathrm{vol})$ is f–parabolic , then the almost soliton is trivial and $(M, \langle\, , \rangle)$ is compact Einstein.*

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Note that the case $\mu = 0$ contains the soliton case, and thus recover a result in [Pigola–R.–Setti, '10] which extends [Petersen–Wylie,'09] where S is either constant or bounded.

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Then $S_- = \max\{-S(x), 0\}$ is a weak solution of

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Computing the lim inf of (4) along $\{x_n\}$ and setting $\overline{\lambda} = \liminf \lambda(x_n)$ shows that

$$\overline{\lambda}S_* - \frac{S_*^2}{m} \geq 0 \quad \Rightarrow \quad \begin{cases} \text{if } \overline{\lambda} = 0 & S_* = 0 \\ \text{if } \overline{\lambda} < 0 & m\overline{\lambda} \leq S_* \leq 0 \\ \text{if } \overline{\lambda} > 0 & 0 \leq S_* \leq m\overline{\lambda}. \end{cases}$$

Since $\lambda_* \leq \overline{\lambda} \leq \lambda^*$, this gives the scalar curvature estimates in (i), (ii) and (iii).

Proof (endpoint cases). Consider e.g. to be in case (iii). Recall that

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- *Myers' theorem* $\Rightarrow$ M compact.
- Soliton equation $\Rightarrow$ $Hess(f) = 0$ $\Rightarrow$ f constant.

□

In the same spirit, we can generalize the scalar curvature estimates obtained in [Case–Shu–Wei,'10] to k–quasi–Einstein manifolds with non–constant scalar curvature. Again, possible rigidity at the endpoints is discussed.

Theorem (R., '10)

Let $(M^m, g_M, e^{-f}\mathrm{dvol})$ be a geodesically complete k–quasi–Einstein manifold, $1 < k < +\infty$, with scalar curvature S and let $S_* = \inf_M S$.

(a) *If $\lambda > 0$, then M is compact and*

$$\frac{m(m-1)}{m+k-1}\lambda < S_* \leq m\lambda. \tag{6}$$

Moreover $S_* \neq m\lambda$ unless M is Einstein.

(b) *If $\lambda = 0$ and $\inf_M f = f_* > -\infty$ then $S_* = 0$. Moreover, either $S > 0$ or $S(x) \equiv 0$. In this latter case, either f is constant (and M is trivial) or M is isometric to the Riemannian product $\mathbb{R} \times \Sigma$ where Σ is a Ricci–flat, totally geodesic hypersurface.*

(c) *If $\lambda < 0$ and $\inf_M f = f_* > -\infty$, then*

$$m\lambda \leq S_* \leq \frac{m(m-1)}{m+k-1}\lambda \tag{7}$$

and $S(x) > m\lambda$ unless M is Einstein.

Rigidity as triviality of an additional structure

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Proof.(Sketch) A computation shows that on a gradient Ricci soliton holds that

$$\frac{1}{2}\Delta_f |\nabla f|^2 = |Hess(f)|^2 - \lambda |\nabla f|^2 \ge -\lambda |\nabla f|^2, \tag{8}$$

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3. $p = 1$: Estimates for the gradient of the potential in [Zhang, '09] $+$ L^1*–Liouville* applied to (9) $\Rightarrow f \equiv const$.

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Theorem (R., '10)

Let $(M^m, g_M, e^{-f}\mathrm{dvol})$ be a complete non-compact k–quasi–Einstein manifold, $1 \leq k < +\infty$. If the q.-E. constant $\lambda \leq 0$, $f \in L^p(M, e^{-\frac{f}{k}}\mathrm{dvol})$ for some $1 < p < +\infty$ and $\inf_M f = f_* > -\infty$, then either $f \equiv const \leq 0$ and M is Einstein or $f > 0$.

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The result reflects on Einstein warped products in this way.

Corollary (R., '10)

Let $N^{m+k} = M^m \times_u F^k$, $k > 1$, be a complete Einstein warped product with ${}^N S \leq 0$, warping function $u(x) = e^{-\frac{f(x)}{k}}$ satisfying $\inf_M f = f_* > -\infty$ and complete Einstein fibre F. Then N is simply a Riemannian product if either f has a local minimum or the base manifold M is complete and non-compact, the warping function satisfies $\int_M |f|^p e^{-\frac{f}{k}}\mathrm{dvol} < +\infty$, for some $1 < p < +\infty$, and $f(x_0) \leq 0$ for some point $x_0 \in M$.

In case M is compact, we recover a recent theorem obtained in [Kim–Kim, '03].

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$$\mathrm{Ric}_f^k \geq c^2 > 0 \quad \overset{[Qian,'96]}{\Longrightarrow} \quad M \text{ compact.}$$

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Following the results we obtained in the classical case in [Mastrolia–R.–Veronelli, '10] (which grow around an idea of E. Calabi), we can allow also some negativity for Ric_f^k.

Indeed, starting from an estimate of the integral of Ric_f^k along minimizing geodesics, an integration by parts shows that the compactness of M^m depends on the behavior, and on the position of the zeros, of the solutions of the differential equation along minimizing geodesics

$$-h'' - \frac{Ric_f^k(\dot{\gamma}, \dot{\gamma})}{m + k - 1}\, h(t) = 0.$$

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Theorem (R.,'10)

Let $\mathrm{Ric}_f^k \geq -(m+k-1)B^2$, *for some constant* $B \geq 0$, $k < +\infty$. *Suppose there is a point* $q \in M$ *such that along each geodesic* $\gamma : [0, +\infty) \rightarrow M$ *parameterized by arc–length, with* $\gamma(0) = q$, *it holds either*

$$\int_a^b t \frac{\mathrm{Ric}_f^k(\dot\gamma, \dot\gamma)}{m+k-1} dt > B\left\{b + a\frac{e^{2Ba}+1}{e^{2Ba}-1}\right\} + \frac{1}{4}\log\left(\frac{b}{a}\right).$$

or

$$\int_a^b t^\alpha \frac{\mathrm{Ric}_f^k(\dot\gamma, \dot\gamma)}{m+k-1} dt > B\left\{b^\alpha + a^\alpha\frac{e^{2Ba}+1}{e^{2Ba}-1}\right\} + \frac{\alpha^2}{4(1-\alpha)}\left\{a^{\alpha-1} - b^{\alpha-1}\right\}$$

for some $0 < a < b$ *and* $\alpha \neq 1$. *Then* M *is compact.*

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If $(M, \langle\, , \,\rangle, e^{-f} d\mathrm{vol})$ is compact and $\mathrm{Ric}_f \geq \lambda > 0$ then by [Fernández-López–García-Río, '08] we have that $|\pi_1(M)| < +\infty$.

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The next theorem extends in the direction of the classical Ambrose theorem these results.

Theorem (Pigola–Rigoli–R.–Setti, '10)

Let $(M, \langle\, , \,\rangle\, , e^{-f}d\mathrm{vol})$ be a complete weighted manifold, $o \in M$ a reference origin. If $\mathrm{Ric}_f \geq 0$ and

$$\int_0^{+\infty} \mathrm{Ric}_f(\dot\gamma, \dot\gamma) = +\infty, \tag{10}$$

for every unit–speed geodesic $\gamma : [0, +\infty) \to M$, $\gamma(0) = o$, then $|\pi_1(M)| < +\infty$.

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Nevertheless hypothesis (10) brings to mind an Ambrose–type condition, it is not completely analogous to this latter since we are also assuming $\mathrm{Ric}_f \geq 0$.
Open problem: can we remove this assumption?

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Thank you!