Rigidity results and topology at infinity of translating solitons of the mean curvature flow

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Weighted manifolds

Let $(M^m, \langle \, , \, \rangle)$ be a complete m-dimensional Riemannian manifold.
Given $f \in C^\infty(M)$, we can consider the weighted manifold $M_f = \left(M, \langle \, , \, \rangle, d\mathrm{vol}_f^M\right)$, where

$$d\mathrm{vol}_f^M = e^{-f} d\mathrm{vol}^M$$

In particular, denoting the geodesic ball of radius R centered at $p \in M$ by $B_r^M(p)$, and by $\partial B_r^M(p)$ its boundary, we set

$$\mathrm{vol}_f(B_r(p)) = \int_{B_r(p)} d\mathrm{vol}_f, \qquad \mathrm{vol}_f(\partial B_r(p)) = \int_{\partial B_r(p)} e^{-f} d\mathrm{vol}_{m-1}.$$

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The f-Laplacian operator on M_f is the diffusion operator defined by

$$\Delta_f = e^f \mathrm{div} e^{-f} \nabla = \Delta - \langle \nabla f, \nabla \rangle .$$

This operator is clearly a symmetric operator in $L^2\left(M, d\mathrm{vol}_f^M\right)$.

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Analytic properties of Δ_f strongly depend on weighted metric-measure properties of the weighted manifold, which in turn are controlled by suitable concepts of curvature adapted to the density of the measure.
The Bakry-Émery Ricci tensor of M_f is defined by

$$\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess}(f).$$

f-minimal hypersurfaces

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Let Σ^m be a connected, orientable m-dimensional Riemannian manifold without boundary smoothly immersed by $x : \Sigma^m \to M_f^{m+1}$ in the weighted manifold M_f^{m+1} and denote by

- $\mathbf{A}$ the second fundamental form of the immersion x, that is

$$\mathbf{A}(X, Y) = (\overline{\nabla}_X Y)^{\perp},$$

  where $\overline{\nabla}$ denotes the Levi-Civita connection on M and $(\cdot)^{\perp}$ denotes the projection on the normal bundle of Σ.
- $\mathbf{H} = tr_{\Sigma}\mathbf{A}$ the mean curvature vector field of Σ.
- H the mean curvature function of Σ, defined with respect to the local Gauss map ν, by $\mathbf{H} = H\nu$.

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Define the <u>weighted area functional</u> of $x : \Sigma^m \to M_f^{m+1}$ by

$$\mathrm{vol}_f(x(\Sigma)) = \int_\Sigma e^{-f} d\mathrm{vol}_\Sigma$$

[Bayle, '03]: letting x_t, $t \in (-\varepsilon, \varepsilon)$, $x_0 = x$, be a smooth compactly supported variation of immersion and denoting by V the associated variational vector field along x, we have that

$$\frac{d}{dt}\mathrm{vol}_f(x_t(\Sigma))\Big|_{t=0} = -\int_\Sigma \langle \mathbf{H}_f, V\rangle\, e^{-f} d\mathrm{vol}_\Sigma.$$

Other characterizations:

- Let $x : \Sigma^m \to (M^{m+1}, g)$ be an immersion. Given $f \in C^\infty(M)$ consider also the conformally changed metric $\widetilde{g} = e^{-\frac{2f}{m}} g$. Then

$$\boxed{(\Sigma, x^*\widetilde{g}) \text{ is minimal in } (M, \widetilde{g})} \quad \Leftrightarrow \quad \boxed{(\Sigma, x^*g) \text{ is } f\text{-minimal in } (M, g)}$$

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- Let $x : \Sigma^m \to (M^m, g_M)$ be an isometric immersion. Let $\mathbb{T} = \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, so that $\mathrm{vol}(\mathbb{T}) = 1$, and consider the warped product $\hat{M}^{m+2} := M^{m+1} \times_{e^{-f}} \mathbb{T} = (M \times \mathbb{T}, g_M + e^{-2f} dt^2)$.
  It was proved in [Smoczyk, '01] that, letting

$$\hat{x} : \hat{\Sigma}^{m+1} := \Sigma \times \mathbb{T} \to (\hat{M}, g_{\hat{M}}),$$

  it holds that

$$\mathbf{H}_{\hat{\Sigma}}(p, t) = (\mathbf{H}_\Sigma(p) + \overline{\nabla} f(p), 0).$$

  In particular

$$\boxed{x : \Sigma \to M_f \text{ is } f\text{-minimal}} \quad \Leftrightarrow \quad \boxed{\hat{x} : \hat{\Sigma} \to \hat{M} \text{ is minimal}}$$

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Let Σ^m be a connected orientable m-dimensional Riemannian manifold without boundary smoothly immersed by $x_0 : \Sigma^m \to \mathbb{R}^{m+1}$ as a complete hypersurface in the Euclidean space $\mathbb{R}^{m+1}$.
We say that $\Sigma_0 = x_0(\Sigma^m)$ is moved along its mean curvature vector if there is a 1-parameter family of smooth immersions $x : \Sigma^m \times [t_0, T) \to \mathbb{R}^{m+1}$, with corresponding hypersurfaces $\Sigma_t = x(\cdot\,,\, t)(\Sigma^m)$, such that it satisfies the following
mean curvature flow initial value problem

$$\text{(MCF)} \qquad \begin{cases} \frac{\partial}{\partial t} x(p,t) = \mathbf{H}(p,t) \\ x(p,\, t_0) = x_0(p), \end{cases}$$

for any $p \in \Sigma^m$, $t \in [t_0, T)$. Here $\mathbf{H}(p,t)$ is the mean curvature vector field of the hypersurface Σ_t at $x(p,t)$.

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A hypersurface Σ is said to be a self-shrinker if the family of surfaces $\Sigma_t = \sqrt{-2t}\Sigma$ is a solution of (MCF). Equivalently, one can simply think of a self-shrinker as an hypersurface whose mean curvature vector field $\mathbf{H}$ satisfies the equation

$$x^\perp = -\mathbf{H}.$$

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   Let $f = \frac{|x|^2}{2}$ and consider the Gaussian space $\mathbb{R}^{m+1}_f$. A simple computation shows that $\overline{\nabla} f = x$, hence self-shrinkers are f-minimal hypersurfaces in the Gaussian space $\mathbb{R}^{m+1}_f$.

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   By a translating soliton for the mean curvature flow (*translator* for short) we mean a connected isometrically immersed complete hypersurface $x : \Sigma^m \to \mathbb{R}^{m+1}$ whose mean curvature vector field $\mathbf{H}$ satisfies the equation

   $$\mathbf{H} = v^{\perp} \tag{T}$$

   for some fixed unit length vector $v \in \mathbb{R}^{m+1}$.
   Solutions of (T) correspond to translating solutions $\{\Sigma_t = \Sigma + tv\}_{t\in\mathbb{R}}$ of the mean curvature flow, and play a key role in the study of slowly forming singularities.

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Let $f = -\langle x, v\rangle$ and consider the weighted manifold $\mathbb{R}^{m+1}_f$. Since $\overline{\nabla} f = -v$, we get that translators are f-minimal hypersurfaces in $\mathbb{R}^{m+1}_f$

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Consider a general ambient Riemannian manifold (M^{m+1}, g_M). Given a smooth vector field X on M, we can consider those solutions of (MCF), with initial data x_0, which move along the integral curves of X.
If X is a conformal vector field on M (i.e. $\mathcal{L}_X g_M = \lambda g_M$, $\lambda \in C^\infty(M)$), a necessary condition for the initial hypersurface $x_0 : \Sigma \to (M^{m+1}, g_M)$ is that its mean curvature vector field satisfies the equation

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These hypersurfaces are said conformal solitons.
Assume now that M is simply connected and that X is also closed, i.e.

$$\overline{\nabla}_Z X = \lambda Z, \quad \forall Z \in TM.$$

Then there exists $f \in C^\infty(M)$ satisfying $\overline{\nabla} f = X$. Hence we get that conformal solitons are f-minimal hypersurfaces Σ in M_f^{m+1}.

Let M_f^{m+1} be a weighted manifold, $x : \Sigma^m \to M_f^{m+1}$ be a f-minimal isometric immersion, and let x_t , $t \in (-\varepsilon, \varepsilon)$, $x_0 = x$, be a smooth compactly supported normal variation of immersions.

We say that the f-minimal hypersurface Σ is f-stable if

$$\frac{d^2}{dt^2}\mathrm{vol}_f(x_t(\Sigma))\Big|_{t=0} \geq 0.$$

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[Bayle, '03]: denote by V the variational vector field along x associated to the variation and let $V = u\nu$, $u \in C_c^\infty$. The second variation formula for the weighted area,

$$\begin{aligned}\frac{d^2}{dt^2}\mathrm{vol}_f(x_t(\Sigma))\Big|_{t=0} &= \int_\Sigma (|\nabla u|^2 - u^2(|\mathbf{A}|^2 + \overline{\mathrm{Ric}}_f(\nu,\nu)))\mathrm{e}^{-f}\, d\mathrm{vol}_\Sigma \\ &= \int_\Sigma u\, L_f\, u\, \mathrm{e}^{-f}\, d\mathrm{vol}_\Sigma,\end{aligned}$$

where $\overline{\mathrm{Ric}}_f$ is the Bakry-Émery Ricci tensor of M_f^{m+1}, and the weighted Jacobi operator L_f is defined by

$$L_f u = -\Delta_f u - (|\mathbf{A}|^2 + \overline{\mathrm{Ric}}_f(\nu,\nu))u.$$

Let $\Omega \subset \Sigma$ be a domain, and consider the weighted Schrödinger operator

$$L_\Omega = -\Delta_f - q(x),$$

$q(x) \in C^{0,\alpha}_{loc}(\Sigma)$, originally defined on $C^\infty_c(\Omega)$. In case $\Omega = \Sigma$ we will simply write $L := L_\Sigma$.

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We say that L_Ω is **semi-bounded** if, for every $u \in C^\infty_c(\Omega)$, the associate quadratic form satisfies

$$\int_\Omega \left(|\nabla u|^2 - qu^2\right) e^{-f} d\mathrm{vol}_\Sigma \geq c \, \|u\|_{L^2(\Omega_f)}$$

for some $c \in \mathbb{R}$. When $c \geq 0$, L_Ω is said to be **non-negative**.

In particular a f-minimal hypersurface Σ in a weighted manifold M_f is f-stable if and only if the weighted Jacobi operator L_f is non-negative.

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If $\Omega \subset\subset \Sigma$ we have that L_Ω is semi-bounded. Hence, denoting its Friedrichs extension with the same symbol, we have that L_Ω has purely discrete spectrum consisting of a divergent sequence of eigenvalues $\{\lambda_k(L_\Omega)\}$.
The first eigenvalue of L_Ω can be written by Rayleigh characterization as

$$\lambda_1(L_\Omega) = \inf_{u \in C^\infty_c(\Sigma),\ u \neq 0} \frac{\int_\Omega (|\nabla u|^2 - qu^2)\, \mathrm{e}^{-f}\, d\mathrm{vol}_\Sigma}{\int_\Omega u^2\, \mathrm{e}^{-f}\, d\mathrm{vol}_\Sigma}.$$

By strict domain monotonicity we have that if $\Omega_1 \subset \Omega_2$, then $\lambda_1(L_{\Omega_2}) < \lambda_1(L_{\Omega_1})$.
The bottom of the spectrum of L on Σ is then defined as

$$\lambda_1^L(\Sigma) = \inf_{\Omega \subset\subset \Sigma} \lambda_1(L_\Omega).$$

Given $\Omega \subset\subset \Sigma$, define the Morse index of L_Ω as

$$Ind(L_\Omega) = \sharp \{\text{negative eigenvalues (counting multiplicities) of } L_\Omega\}.$$

Again by strict domain monotonicity, we have that if $\Omega_1 \subset \Omega_2$, then $Ind^{L_{\Omega_1}} < Ind^{L_{\Omega_2}}$.

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We define the f-index of a complete f-minimal hypersurface as the generalized Morse index of the weighted Jacobi operator L_f on Σ. Namely,

$$Ind_f(\Sigma) := Ind^{L_f}(\Sigma) = \sup_{\Omega \subset\subset \Sigma} Ind^{L_f}(\Omega).$$

Geometrically, the f-index of Σ can be described as the maximum dimension of the linear space of compactly supported deformations that decrease the weighted area up to second order.

PDE counterpart of spectral properties:

- Weighted version of [Fischer–Colbrie-Schoen, '80], [Moss-Piepenbrink, '78]: the positivity of a (weighted) Schrödinger operator can be formulated in term of the existence of positive solutions of the associated linear equation. **For any domain** $\Omega \subset \Sigma$:

$$\boxed{\lambda_1^L(\Omega) \geq 0} \quad \Leftrightarrow \quad \boxed{\begin{array}{l}\exists\, \varphi > 0 \text{ solution} \\ \text{of } L\varphi = 0 \text{ on } \Omega\end{array}}$$

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- [Fischer–Colbrie, '85], [Devyver, '12]: the finiteness of the Morse index of a (weighted) Schrödinger operator can be formulated in terms of the existence of a positive solution of the associated linear equation **outside a compact set**.

$$\boxed{Ind^L(\Sigma) < +\infty} \quad \Leftrightarrow \quad \boxed{\begin{array}{l}\lambda_1^L(\Sigma\setminus\Omega) \geq 0, \text{ for some} \\ \Omega \subset\subset \Sigma\end{array}} \quad \Leftrightarrow \quad \boxed{\begin{array}{l}\exists\,\varphi > 0 \text{ solution} \\ \text{of } L\varphi = 0 \text{ on } \Sigma\setminus\Omega, \text{ for some} \\ \Omega \subset\subset \Sigma\end{array}}$$

Examples:

1. *Minimal hypersurfaces tangent to* v.

   If $\mathbf{H} = 0$, by (T) we have that v must be tangential to the translator. Consequently, translators which are also minimal hypersurfaces **split** as the product

$$\tilde{\Sigma} \times L,$$

   where L is a line parallel to v and $\tilde{\Sigma}$ is a minimal hypersurface in $L^{\perp}$.

Examples:

1. *Minimal hypersurfaces tangent to v.*

2. *Grim reaper and grim reaper cylinder.*

   The **only possible translating curve with non-identically zero curvature** is given by the graph of
   $$y = -\log(\cos(x)),$$
   where $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. This curve is called the **grim reaper**, [Grayson, '87].

   Taking the orthogonal product of a grim reaper with $\mathbb{R}^{m-1}$, one easily obtains another (higher dimensional) example of translator in $\mathbb{R}^{m+1}$, the **grim reaper cylinder**.
   It can be easily proved that these examples are mean convex. Indeed, they have only one strictly positive principal curvature.

Translators for the MCF: $\mathbf{H} = v^{\perp}$

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2. *Grim reaper and grim reaper cylinder.*
3. *Bowl soliton and the translating catenoid.*

   [Altschuler-Wu, '94]: there exist a unique (up to rigid motion) solution of (T) which is **rotationally symmetric and strictly convex**.
   When $m = 1$ this is the grim reaper. When $m > 1$, the solution (which is actually an entire graph growing quadratically at infinity) roughly looks like a paraboloid, and is usually called the **bowl soliton**.

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   [Wang, '11]: in dimension $m = 2$ any **entire convex** translator must be rotationally symmetric, and hence the bowl soliton. For $m \geq 3$ there exist entire strictly convex translators that are **not rotationally symmetric**.

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   [Clutterbuck-Schnürer-Schulze, '07]: rotationally symmetric translators coincide (up to rigid motion) either with the bowl soliton or with the **translating catenoid**, a complete non-convex translator made up of the union of two graphical "winglike" solutions in the complement of a ball that are asymptotic at infinity to a bowl soliton.

Translators for the MCF: $\mathbf{H} = v^{\perp}$

Examples:

1. *Minimal hypersurfaces tangent to* v.
2. *Grim reaper and grim reaper cylinder*.
3. *Bowl soliton and the translating catenoid*.
4. 2*-dimensional translators with infinite genus or prescribed finite genus*.

   Tool: **desingularization technique**. See [Nguyen, '09], [Nguyen, '13], [Smith, '15].

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$$L_f = -\Delta_f - |\mathbf{A}|^2.$$

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Proof. A computation shows that on a translator holds that

$$\Delta_f H = -|\mathbf{A}|^2 H.$$

Strong maximum principle for Δ_f $\Rightarrow$ either H never vanishes, or $H \equiv 0$.
In the first case, by the weighted version of [Fischer–Colbrie-Schoen, '80], we get f-stability.

□

We can prove the following rigidity result for f-stable translators under a weighted L^2 condition on the norm of the second fundamental form.

Theorem (Impera-R., '14)

Let $x : \Sigma^m \to \mathbb{R}^{m+1}$ be a f-stable translator of the mean curvature flow. Assume that $|\mathbf{A}| \in L^2(\Sigma_f)$. Then Σ is a translator hyperplane.

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On the other hand, under this assumption we are able to strengthen the result as follows.

Theorem (Impera-R., '14)

Let $x : \Sigma^m \to \mathbb{R}^{m+1}$ be a translator with mean curvature which does not change sign. Suppose that the traceless second fundamental form $\mathbf{\Phi} = \mathbf{A} - \frac{H}{m}\mathbf{Id}$ satisfies $|\mathbf{\Phi}| \in L^2(\Sigma_f)$. Then Σ is a translator hyperplane.

Analytical tools:

- f-vanishing result; [R., '14], adapting to the weighted setting [Pigola-Rigoli-Setti, '05].

On M_f let the functions $u \geq 0$, $v > 0$ satisfy

(1) $$\Delta_f u + a(x)u \geq 0,$$

(2) $$\Delta_f v + \delta a(x)v \leq 0$$

for some constant $\delta \geq 1$ and $a(x) \in C^0(M)$. Assume that $u \in L^{2\beta}(M_f)$, $1 \leq \beta \leq \delta$.

$\Rightarrow$

There exists a constant $C \geq 0$ such that $u^\delta = Cv$. Furthermore,

(i) If $\delta > 1$ then u is constant on M and either $a \equiv 0$ or $u \equiv 0$;

(ii) If $\delta = 1$ and $u \not\equiv 0$, v and therefore u^δ satisfy (2) with equality sign.

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On M_f let the functions $u \geq 0$, $v > 0$ satisfy (1) $\Delta_f u + a(x)u \geq 0,$ (2) $\Delta_f v + \delta a(x)v \leq 0$ for some constant $\delta \geq 1$ and $a(x) \in C^0(M)$. Assume that $u \in L^{2\beta}(M_f)$, $1 \leq \beta \leq \delta$.	$\Rightarrow$	There exists a constant $C \geq 0$ such that $u^\delta = Cv$. Furthermore, (i) If $\delta > 1$ then u is constant on M and either $a \equiv 0$ or $u \equiv 0$; (ii) If $\delta = 1$ and $u \not\equiv 0$, v and therefore u^δ satisfy (2) with equality sign.

- preliminary lemma; [Impera-R., '14].

Let $x : \Sigma^m \to \mathbb{R}^{m+1}$ be a f-stable translator with $H(p) \neq 0$ for some $p \in \Sigma$, and $\lvert\mathbf{A}\rvert \in L^2(\Sigma_f)$. Let $\omega \in C^2(\Sigma)$ be a positive solution of the stability equation $\Delta_f \omega + \lvert\mathbf{A}\rvert^2 \omega = 0.$	$\Rightarrow$	There exists a constant $C \in \mathbb{R} \setminus \{0\}$ such that $H = C\omega$. In particular $\lvert H\rvert > 0$.

Proof of the rigidity theorem. (Sketch) On a translator we have the validity of the following Simons' type formula

$$|\mathbf{A}|\left(\Delta_f + |\mathbf{A}|^2\right)|\mathbf{A}| = |\nabla \mathbf{A}| - |\nabla |\mathbf{A}||^2 \geq 0. \tag{3}$$

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$$\left(1 + \frac{2}{m+1}\right)|\nabla |\mathbf{A}|| \leq |\nabla \mathbf{A}|^2 + \frac{2m}{m+2}|\nabla H|^2,$$

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- If $\exists\, p \in \Sigma$ s.t. $H(p) \neq 0$ $\overset{\text{Preliminary lemma}}{\Rightarrow}$ $|H| > 0$ on Σ, $\omega = C_2 H,\ C_2 \in \mathbb{R} \setminus \{0\}$. $\Rightarrow$ $\boxed{|\mathbf{A}| = C_3 H}$

Proof. (continued) Using these two identities, we can proceed as follows.

- **Case 1**: $\exists\, p$ s.t. $rk(\mathbf{A}_p) \geq 2 \quad \Rightarrow \quad \nabla \mathbf{A} \equiv 0$ on Σ

  $\overset{[\text{Lawson}, '69]}{\Rightarrow}$ Σ is a cylindrical product $\mathbb{S}^k \times \mathbb{R}^{m-k}$ for some $k = 0, \dots, m-1$. Contradiction.

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- **Case 2**: $rk(\mathbf{A}) \equiv 1 \quad \Rightarrow$ Σ is invariant under the isometric translations in the $(m+1)$-dimensional subspace P spanned by a global o.n. frame for $Ker(\mathbf{A})$.

  $\Sigma \cong \Gamma \times P$, with $\Gamma \subset \mathbb{R}^2$ translator curve with $|H| > 0 \quad \Rightarrow \quad \Sigma$ is a grim reaper cylinder. Contradiction with the assumption $|\mathbf{A}| \in L^2(\Sigma_f)$.

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Topology at infinity of translators

Get information on the number of ends of f-stable translators and translators with finite f-index.

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Recall that an end E of M_f w.r.t. $D \subset\subset M$, ∂D smooth, is said to be f-parabolic if and only every positive f-superharmonic function u satisfying $\partial u/\partial n \geq 0$ on ∂E, n being the unit outward normal to ∂E, is constant. Otherwise the end will be called non-f-parabolic.

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Main tool: **Weighted version of Li-Tam theory**; see [Impera-R. '13].

- Given $D \subset\subset M_f$, the number $N(D)$ of non-f-parabolic ends w.r.t. D satisfies

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  where $\mathcal{H}^{\infty}_{\mathcal{D}}(M_f)$ denotes the space of bounded f-harmonic functions with finite Dirichlet weighted integral on M_f.
- In particular one proves that if M_f has at least two non-f-parabolic ends, then it supports a non-constant bounded f-harmonic function with finite Dirichlet weighted integral.

Using Smoczyk's correspondence between translators and minimal hypersurfaces in $\mathbb{R}^{m+1} \times_{e^{-f}} \mathbb{T}^1$, we obtain that, on a translator Σ, for every $h \in C_c^\infty(\Sigma)$, $h \geq 0$,

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Proof. (Sketch) Assume by contradiction that Σ has at least two non-f-parabolic ends.

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- Fix $p \in \Sigma$ and let $\{E_i(p)\}_{i=1}^m$ be an o.n. basis for $T_p\Sigma$ such that $E_1(p) = \frac{\nabla u}{|\nabla u|}(p)$. Hence, at p, $\mathrm{tr}(\mathrm{Ric}_f) = -|\mathbf{A}|^2 - \sum_{i=2}^m |\mathbf{A}E_i|^2$.

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- Note that this in particular applies to translators with $|H| > 0$. Moreover, in case $H \equiv 0$, i.e. $\Sigma = \tilde{\Sigma} \times \mathbb{R}$ with $\tilde{\Sigma}$ minimal, Σ has only one end.

Proof. (Sketch) Assume by contradiction that Σ has at least two non-f-parabolic ends.

Weighted Li-Tam theory $\Rightarrow$ $\exists\, u \in \mathcal{H}^{\infty}_{\mathcal{D}}(\Sigma_f)$, $u \neq const.$

Bochner formula for f-harmonic functions
Lower bound for Ric_f on a translator $\Rightarrow$ $\boxed{\Delta_f |\nabla u| + |\mathbf{A}|^2 |\nabla u| \geq 0}$ $(\#)$
Kato inequality

f-stability $\Rightarrow$ $\exists\, v > 0$ s.t. $\boxed{\Delta_f v + |\mathbf{A}|^2 v = 0}$

f-vanishing $\Rightarrow$ either $|\nabla u| \equiv 0$ (contradiction) or $|\nabla u| = Cv$ for some positive constant C and $|\nabla u|$ satisfies $(\#)$ with the equality sign. In particular we get that $\mathrm{Ric}_f(\nabla u, \nabla u) = -|\mathbf{A}|^2 |\nabla u|^2$.

- Fix $p \in \Sigma$ and let $\{E_i(p)\}_{i=1}^m$ be an o.n. basis for $T_p\Sigma$ such that $E_1(p) = \frac{\nabla u}{|\nabla u|}(p)$. Hence, at p, $\mathrm{tr}(\mathrm{Ric}_f) = -|\mathbf{A}|^2 - \sum_{i=2}^m |\mathbf{A} E_i|^2$.
- On the other hand, since $\Delta_f f = -1 \overset{\mathrm{Gauss}'\,\mathrm{eqn.}}{\Rightarrow} \mathrm{tr}(\mathrm{Ric}_f) = -|\mathbf{A}|^2$.

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$\Rightarrow$ $|\mathbf{A}|^2 = H^2$ on Σ. $\overset{\mathrm{Gauss'eqn.}}{\Rightarrow}$ $R \equiv 0$.

Using a result in [Martín-Savas–Halilaj-Smoczyk, '14], we get that Σ has to be either a grim reaper cylinder or a translator hyperplane. Contradiction.

The case $Ind_f(\Sigma) < +\infty$

Analytical tool:

- Weighted version of an abstract finiteness result obtained in [Pigola-Rigoli-Setti, '08].

Let E be a Riemannian vector bundle of rank l over M_f. Denote by $\Gamma(E)$ the space of its smooth sections. Let $a(x) \in C^0(M)$, β, δ constants s.t. $1 \leq \beta \leq \delta$.

Let $V = V(a, f, \beta, \delta) \subset \Gamma(E)$ be any vector space with the following two properties.

(i) Every $\xi \in V$ has the unique continuation property,

(ii) For any $\xi \in V$, the locally Lipschitz function $u = |\xi|$ satisfies

$$\begin{cases} u(\Delta_f u + a(x)u) \geq 0 \quad \text{weakly on } M \\ u \in L^{2p}(M_f). \end{cases}$$

If $\exists\, v > 0$ s.t. $\Delta_f v + \delta a(x) v \leq 0$ weakly outside $K \subset\subset M$.

$\Rightarrow$ $\boxed{\dim V < +\infty}$

Let $\delta_f = \delta + i_{\nabla f}$ and denote with $\Delta_H^f = \delta_f d + d\delta_f$ the Hodge f-Laplacian on M_f. Set

- $\mathcal{H}^1(M_f) = \{1\text{−forms}\, \omega \text{ s.t. } \Delta_H^f \omega = 0\}$
- $V = L^{2,f}\mathcal{H}^1(M_f) = \{\xi \in \mathcal{H}^1(M_f) \text{ s.t. } |\xi| \in L^2(M_f)\}$.

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Then

(i) the equation $\Delta_H^f\xi = 0$ is equivalent to the equation $\Delta_H\xi = F(x, \xi, d\xi)$ with F satisfying the structural conditions of Aronszajn-Cordes. Hence every $\xi \in \mathcal{H}^1(M_f)$ has the unique continuation property.

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(ii) The following f-Weitzenbock formula for 1-forms $\xi \in \mathcal{H}^1(M_f)$ holds

$$\frac{1}{2}\Delta_f |\xi|^2 = |D\xi|^2 + \mathrm{Ric}_f(\xi^\sharp, \xi^\sharp).$$

Let $\mathrm{Ric}_f \geq -a(x)$, $a(x) \in C^0(M)$.
Using Kato inequality, we get that, for any $\xi \in V$, the function $u = |\xi|$ satisfies

$$\begin{cases} u(\Delta_f u + a(x)u) \geq 0 \ \text{ weakly on } M \\ \int_M u^2 e^{-f} d\mathrm{vol}_M < +\infty. \end{cases}$$

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Moreover, letting $L = -\Delta_f - a(x)$, if $\mathrm{Ind}^L(M) < +\infty$ we have that there exists a solution $v > 0$ of the differential inequality $\Delta_f v + a(x)v \leq 0$ weakly outside $K \subset\subset M$.

Weighted version of the abstract finiteness theorem

$\Downarrow$

Let M_f be a complete non-compact weighted manifold satisfying

$$\mathrm{Ric}_f \geq -a(x)$$

for some continuous function $a(x)$, and let $L = -\Delta_f - a(x)$. If $\mathrm{Ind}^L(M_f) < +\infty$, then

$$\dim L^{2,f}\mathcal{H}^1(M_f) < +\infty$$

Weighted version of the abstract finiteness theorem

The number $N(D)$ of non-f-parabolic ends of M w.r.t. any relatively compact domain D satisfies $N(D) \leq \dim \mathcal{H}^{\infty}_{\mathcal{D}}(M_f)$.

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Theorem (Impera-R., '14)

Let $x : \Sigma^m \to \mathbb{R}^{m+1}$, $m \geq 2$, be a translator with finite f-index. Then Σ has finitely many ends.

Thank you!

References

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