Extremals of Log Sobolev inequality on non-compact manifolds and Ricci soliton structures

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Joint work with Giona Veronelli

Introduction

A **Ricci soliton structure** on a Riemannian manifold (M^m, g) is the choice of a vector field X (if any) such that

(RS) $$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda_S g,$$

for some constant $\lambda_S \in \mathbb{R}$.

$\lambda_S < 0$: *expanding* soliton $\lambda_S = 0$: *steady* soliton $\lambda_S > 0$: *shrinking* soliton

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Why study Ricci solitons?

- Natural extensions of Einstein manifolds: equation (RS) reduces to the Einstein equation when X is a Killing vector field.
- Self-similar solutions to the Ricci flow;
- Singularity models of the Ricci flow: they arise as blow-up limits of the Ricci flow when singularities develop.

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Can the soliton vector field X always be replaced by a gradient vector field (i.e. $X = \nabla f + Y$ for some $f \in C^{\infty}(M)$ and Y Killing)?

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Theorem (Ivey, '93, Hamilton, '88, Perelman, '02)

Every non-trivial compact shrinking Ricci soliton supports a shrinking gradient Ricci soliton structure.

Complete non-compact setting

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- *Gaussian and anti-Gaussian soliton*: $(\mathbb{R}^m, g_{\mathbb{R}^m}, \nabla f)$

$$f(x) = \frac{1}{2}\lambda_S |x|^2 + g_{R^m}(x, a) + b, \qquad a \in \mathbb{R}^m,\ b \in \mathbb{R}.$$

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- Riemannian products $(\mathbb{R}^n \times N^k, g_{\mathbb{R}^n} + g_N, \nabla f)$
  (N^k, g_N) Einstein with Einstein constant $\lambda_S \neq 0$;
  $f : \mathbb{R}^m \times N^k \to \mathbb{R}$ given by

$$f(x, p) = \frac{\lambda_S}{2} |x|^2_{\mathbb{R}^n} + g_{\mathbb{R}^n}(x, a) + b, \qquad a \in \mathbb{R}^n,\ b \in \mathbb{R}.$$

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- *Hamilton's Cigar soliton*: $\left(\mathbb{R}^2, \frac{dx^2+dy^2}{1+x^2+y^2}, \nabla f\right)$,

$$f(x,y) = -ln(1 + x^2 + y^2).$$

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- *Bryant soliton*: unique (up to homothety) complete rotationally symmetric steady gradient Ricci soliton on $\mathbb{R}^m$ for $m \geq 3$.

- [Baird-Danielo, '07]: Complete description of the soliton structures on the eight three-dimensional geometries of Thurston.
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Can we obtain the same conclusion under different curvature conditions?

Log Sobolev functional

Let $(M^{m\geq 3}, g)$ be a complete connected Riemannian manifold, $\Omega \subset M$ a domain in M, and denote by R the scalar curvature and by $d\mathrm{vol}$ the Riemannian volume measure.

The **Log Sobolev functional** that we deal with is just the usual one perturbed by the scalar curvature of the manifold i.e

$$\mathcal{L}(v, \Omega, g) := \int_\Omega \left(4|\nabla v|^2 + \mathrm{R}v^2 - v^2 \ln v^2\right) d\mathrm{vol}, \qquad v \in W^{1,2}(\Omega).$$

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We define the **best Log Sobolev constant of the domain** $\Omega \subset M$ as

$$\lambda(\Omega) = \inf \left\{ \int_\Omega \left[4|\nabla v|^2 + Rv^2 - v^2 \ln v^2\right] d\mathrm{vol} \text{ s.t. } v \in C_c^\infty(\Omega);\ \|v\|_{L^2(\Omega)} = 1 \right\}.$$

When $\Omega = M$, we will denote by $\lambda := \lambda(M)$ the **best Log Sobolev constant of** (M, g).

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Suppose that $\lambda(\Omega) > -\infty$. A function $v \in W^{1,2}(\Omega)$ is called an **extremal of the Log Sobolev functional** $\mathcal{L}$ on Ω, if $\|v\|_{L^2(\Omega)} = 1$ and

$$\int_{\Omega} \left(4|\nabla v|^2 + \mathrm{R} v^2 - v^2 \ln v^2 \right) d\mathrm{vol} = \lambda(\Omega).$$

The **Euler-Lagrange equation** for this constrained variational problem is given by

(EL) $$4\Delta v - \mathrm{R}v + 2v \ln v + \lambda v = 0$$

Here, and from the point onward, we will implicitly assume that $v \geq 0$ when $\ln v$ appears and that $v \ln v(x) = 0$ when $v(x) = 0$.

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The existence problem for extremals of the Log Sobolev functional in the compact case is completely settled by the following

Theorem (Rothaus, '81)

Let $\Omega \subset\subset M$ be a relatively compact domain in M with smooth boundary $\partial\Omega$. Then

1. $\lambda(\Omega) > -\infty$;
2. $\exists$ v *smooth positive extremal of $\mathcal{L}$ on (Ω, g), taking the boundary values zero continuously.*

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- Note that for the validity of the first part it suffices that the scalar curvature R on (Ω, g) is bounded from below, and an L^2-Sobolev inequality holds on (Ω, g).

Existence of a gradient Ricci soliton structure: elliptic approach

[Eminenti-La Nave-Mantegazza, '08]

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Let (M^m, g, X) be a compact shrinking Ricci soliton satisfying (RS).
Note that, for a generic smooth function $f : M \to \mathbb{R}$, one can compute that

$$e^f div(2e^{-f}(\mathrm{Ric} + \mathrm{Hess}(f) - \lambda_S g)) = \nabla(\mathrm{R} + 2\Delta f - |\nabla f|^2 + 2\lambda_S f).$$

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[Rothaus, '81] $\Rightarrow$

$\exists\, v > 0 \in C^\infty(M)$ on the rescaled manifold $(M, 2\lambda_S g)$ s.t.

$$4\Delta_{2\lambda_S g} v - R_{2\lambda_S g} v + 2v \ln v + \lambda(M, 2\lambda_S g) v = 0.$$

By a conformal change and setting $f = -\frac{m}{2}\ln(4\pi) - 2\ln v$, we get that f satisfies

$$2\Delta_g f + R - |\nabla f|^2 + 2\lambda_S f = 2\lambda_S \left[\lambda(M, 2\lambda_S g) - \frac{m}{2}\ln(4\pi)\right] = \mathrm{const}.$$

Hence, for this choice of f, we have

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Let $\boxed{T = i_{\nabla f - X}(\mathrm{Ric} + \mathrm{Hess}(f) - \lambda_S g)}$. Using (1) and (RS) one can compute that

$$e^f div(e^{-f} T) = |\mathrm{Ric} + \mathrm{Hess}(f) - \lambda_S g|^2 \geq 0,$$

and, using the divergence theorem, we get the conclusion.

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   We would like to use a global divergence theorem à la Gaffney-Karp, [Karp, '81].

> Let (M, g) be a complete Riemannian manifold, $f \in C^\infty(M)$, $|Y| \in L^1_{loc}(M)$, s.t. for some $o \in M$
> $$\int_{B_{2R}(o)\setminus B_R(o)} |Y|e^{-f} = o(R) \qquad R \to \infty.$$
> If either $(\mathrm{div}(e^{-f}Y))_- \in L^1(M)$ or $(\mathrm{div}(e^{-f}Y))_+ \in L^1(M)$, then $\int_M \mathrm{div}(e^{-f}Y) = 0$.

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In this order of ideas we need:

- **decaying estimate at infinity for the extremal**.
- **control on the growth at infinity of the soliton field X**.

Theorem (Zhang, '12)

(a) *Let (M^m, g) be a complete (connected) non-compact Riemannian manifold with* **bounded geometry** *and suppose the validity of the* **condition at infinity**

$$\lambda < \lambda_\infty := \liminf_{r\to\infty} \lambda(M \setminus B_r(o)).$$

Then there exists a smooth extremal for $\mathcal{L}$ on M.

(b) *There exists a complete non-compact manifold with bounded geometry such that $\lambda = \lambda_\infty$, but $\mathcal{L}$ does not have an extremal.*

Existence of extremals of $\mathcal{L}$ on non-compact manifolds

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Bounded geometry:

(i) The Riemann curvature tensor and all its covariant derivatives are bounded.

(ii) $\exists r_0 > 0$ s.t. $\inf_{x\in M} \mathrm{vol}(B_{r_0}(x)) > 0$.

By [Cheeger-Gromov-Taylor, '82] these two properties imply

(iii) $\exists\, i > 0$ positive lower bound on the injectivity radii on (M, g).

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Condition at infinity:

- Avoids the escape at ∞ of the $\mathcal{L}$-content of a minimizing sequence.
- Ex. (M, g) asimptotically Euclidean, admitting a domain $D \subset\subset M$ s.t. $\lambda(D) < \lambda(\mathbb{R}^m) = \frac{m}{2}\ln(4\pi) - m$. Then $\lambda \leq \lambda(D) < \lambda_\infty = \lambda(\mathbb{R}^m)$. E.g. let D be the scaled flat tube $h^2(\mathbb{S}^1 \times \mathbb{S}^1) \times [A, B]$. Then $\lambda(D) \to -\infty$ as $h \to 0$.

Theorem (R.-Veronelli, '16)

Let (M^m, g) be a (connected) complete non-compact Riemannian manifold s.t,

$$\mathrm{Ric} \geq -(m-1)K \qquad \text{and} \qquad \mathrm{inj}_{(M,g)} \geq i_0 > 0,$$

for some $K \in [0, +\infty)$, $i_0 \in \mathbb{R}^+$. If $\lambda < \lambda_\infty$, then there exists a smooth extremal for $\mathcal{L}$.

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Proof outline

1. Given $o \in M$, let $D(o,k) \subset\subset M$ be an exhaustion of M s.t. $\partial D(o,k)$ is smooth for any $k \in \mathbb{N}$ and let $\lambda_k := \lambda(D(o,k))$.

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2. $\exists\, v_k$ smooth positive extremal on $D(o,k)$ in $W^{1,2}(D(o,k)) \cap C^0(\overline{D(o,k)})$ s.t.

$$\begin{cases} 4\Delta v_k - R v_k + 2 v_k \ln v_k + \lambda_k v_k = 0, & \text{in} \quad D(o,k) \\ v_k = 0, & \text{on} \quad \partial D(o,k) \\ \|v_k\|_{L^2(D(o,k))} = 1. & \end{cases}$$

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3. Extend v_k by 0 on $M \setminus D(o,k)$. Then $\mathcal{L}(v_k, M, g) = \lambda_k \searrow \lambda$.

Proof outline (continued)

4. $\{v_k\}$ is uniformly bounded in $W^{1,2}(M)$.

   Hence there exists $v \in W^{1,2}(M)$ such that $v_k \to v$ weakly in $W^{1,2}(M)$, strongly in L^p on compact sets for every $p \in \left(1, \frac{2m}{m-2}\right)$, and a.e. in M.

   In particular $v \in W^{1,2}(M) \cap L^2_{\mathrm{loc}}(M)$, $v \geq 0$ a.e. in M and, since $\int_A v_k^2 d\mathrm{vol} \leq 1$ for every $A \subset M$ compact, the same holds for v, and hence $\int_M v^2 d\mathrm{vol} \leq 1$.

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6. Let $x \in M$, w bounded solution to (EL) in $B_2(x)$ s.t. $\|w\|_{L^2(B_2(x))} \leq 1$.
   Then $\exists\, C = C(m, K, i_0, \lambda, \sup_{B_1(x)} |\nabla R|) > 0$ s.t.

$$\sup_{B_1(x)} w^2 \leq C \int_{B_2(x)} w^2 d\mathrm{vol};$$

$$\sup_{B_{1/2}(x)} |\nabla w|^2 \leq C \int_{B_1(x)} w^2 d\mathrm{vol}.$$

Proof outline (end)

7. Consider $E_i \nearrow M$, $E_i \subset\subset E_{i+1}$, ∂E_i smooth, s.t. $\forall\, i$ there exists K_i s.t. $B_2(E_i) \subset D(0,k)$ for $k \geq K_i$.
   Then $\{v_k\}_{k>K_j}$ is bounded in $C^1(E_j)$ and, up to a subsequence, $v_k \to v$ on E_j in $C^{0,\alpha}$, $0 < \alpha < 1$. Thus $0 \leq v \in C^{0,\alpha}_{\mathrm{loc}}(M)$ and $\exists\, x \in M$ s.t. $v(x) > 0$.

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10. Integrating (EL) and proving that indeed $\int_M v^2 = 1$, we eventually get that v is an extremal for $\mathcal{L}$.

□

The first key tool to deal with the final integration by parts in the "elliptic proof" is the following

Theorem (R.-Veronelli, '16)

Let (M^m, g) be a complete Riemannian manifold such that

(Hp) $$|\mathrm{Ric}| \leq (m-1)K \qquad \text{and} \qquad \mathrm{inj}_{(M,g)} \geq i_0 > 0.$$

Let $o \in M$ and u be a bounded subsolution to (EL) *on M such that $\|u\|_{L^2(M)} \leq 1$. Then there exist positive numbers r_0, a and A, which may depend on K, i_0 and the location of o such that*

$$u(x) \leq Ae^{-ad^2(x,o)},$$

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This is proved comparing u with a suitable model function constructed starting from a **distance-like function with a uniform control on the gradient and the Hessian** on (M, g).

Proposition (R.-Veronelli, '16)

Given $m \geq 2$, $K \in [0, \infty)$, *there exists a constant* $C_{m,K} \in (1, \infty)$, *depending only on* m *and* K, *such that if* (M^m, g) *is a complete non-compact Riemannian manifold satisfying* (Hp) *and* $o \in M$, *then there exists* $h \in C^\infty(M)$ *such that*

$$d(x, o) + 1 \leq h(x) \leq d(x, o) + C_{m,K} \tag{2}$$

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Proof.(Idea) (coming from [Tam, '10]):

- By [Greene-Wu, '76] we know that given any complete Riemannian manifold (M, g) and $o \in M$ there exists a $u \in C^\infty(M)$ with

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- **Evolve u by the heat equation to have also a uniform bound on the Hessian**. Namely, let $H : M \times M \times (0, \infty) \to (0, \infty)$ be the heat kernel of (M, g). Then the function $h : M \times (0, \infty) \to \mathbb{R}$ defined by

$$h(x,t) := \int_M H(x,y,t) u(y) d\mathrm{vol}(y)$$

is a solution to the heat equation with $\lim_{t \to 0} h(x,t) = u(x)$ uniformly in $x \in M$. Using our assumptions it is possible to prove that $h(x) := h(x,1)$ satisfies conditions (2), (3), (4) of the statement.

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For any $n \geq 1$, let $\phi_n \in C^\infty([0,+\infty))$ be the cut-off defined by $\phi_n(t) := \phi(t/n - 1)$. In particular

$$\phi_n|_{[0,n]} = 1, \quad \phi_n|_{[2n,\infty)} = 0, \qquad |\phi_n'| \leq 2/n, \quad |\phi_n''| \leq a/n^2.$$

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Let the constant $C_{m,K}$ and the function $h \in C^\infty(M)$ be as in the previous Proposition.

For each integer $n > C_{m,K}$, define $\chi_n := \phi_n \circ h$. Then

- $\chi_n = 1$ on $B_{n-C_{m,K}}(o)$
- $\mathrm{supp}(\chi_n) \subset B_{2n-1}(o)$.
- $\|\nabla \chi_n\|_\infty \to 0$ as $n \to \infty$
- $\|\mathrm{Hess}(\chi_n)\|_\infty \to 0$ as $n \to \infty$.

Growth of the radial part of the soliton field when Ric is bounded

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Let (M, g, X) be a complete non-compact Ricci soliton i.e.

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda_S g$$

for some $\lambda_S \in \mathbb{R}$.
Then for any unit-speed geodesic $\gamma : [0, L] \to M$ we have that

$$\begin{aligned}
\frac{d}{dt}\, g(\dot\gamma(t), X(\gamma(t))) &= g(\nabla_{\dot\gamma}\dot\gamma(t), X(\gamma(t))) + g(\dot\gamma(t), \nabla_{\dot\gamma} X(\gamma(t))) \\
&= \frac{1}{2}\mathcal{L}_X g\,(\dot\gamma(t), \dot\gamma(t)) \\
&= (\lambda_S g - \mathrm{Ric})(\dot\gamma(t), \dot\gamma(t)).
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Suppose that $|\mathrm{Ric}| \leq (m-1)K$ for some constant $K \geq 0$. Then

$$\boxed{\left|\tfrac{d}{dt}\, g(\dot\gamma(t), X(\gamma(t)))\right| \leq |\lambda_S| + (m-1)K.}$$

In particular

$$\boxed{|g(\dot\gamma(L), X(\gamma(L))) - g(\dot\gamma(0), X(\gamma(0)))| \leq L(|\lambda_S| + (m-1)K).}$$

Idea to estimate $|X|$:

- By continuity $|X|$ is bounded on the unitary geodesic ball $B_1(o) \subset M$.
- Apply the previous estimate along all the geodesics γ_y connecting $y \in B_1(o)$ to $q \in M$.
- If we're able to prove that the family of vectors $\{\dot{\gamma}_y(q)\}_{y \in B_1(o)}$ covers an angle in T_qM that is large enough, then we can obtain a quantitative control of $|X|(q)$ by means of the control on the $|g(\dot{\gamma}_y, X)|$.

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Lemma (R.-Veronelli, '16 - "Ricci hinge Lemma")

Let (M^m, g) be complete s.t. $\mathrm{Ric} \geq -(m-1)K$ for some $K \geq 0$. Let o, q be points of M with $d(o,q) = r > 1$. Let $W \in S_qM$, with S_qM denoting the set of unitary vectors in T_qM. Then there exists $Z \in S_qM$ such that $\exp_q(sZ) \in B_1(o)$ for some $s \in [r-1, r+1]$ and

$$|g(Z,W)| \geq \begin{cases} C_0 r^{1-m}, & \text{if } K = 0, \\ C_1 e^{-(m-1)\sqrt{K}r}, & \text{if } K > 0, \end{cases} \tag{5}$$

for some explicit constants C_0, C_1 depending only on m, K and $\mathrm{vol}(B_1(o))$.

Theorem (R.-Veronelli, '16)

Let (M^m, g, X) *be a complete non-compact Ricci soliton satisfying* (RS) *for some* $\lambda_S \in \mathbb{R}$*. Suppose that* $|\mathrm{Ric}| \leq (m-1)K$ *for some constant* $K \geq 0$*. For any reference point* $o \in M$ *there exists a positive constant* $C > 0$*, depending on* m*,* K*,* λ *and on*

$$X^* := \max_{y \in \overline{B_1(o)}} |X(y)|,$$

such that for all $q \in M$ *it holds*

$$|X|(q) \leq \begin{cases} Cd(q,o)^m & \textit{if } K = 0 \\ Cd(q,o)e^{(m-1)\sqrt{K}d(q,o)} & \textit{if } K > 0. \end{cases}$$

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Remarks

- For gradient Ricci solitons it was proved in [Zhang, '09] that the whole norm of the soliton field grows at most linearly. Actually in the generic setting we are not able to get such a linear bound for $|X|$, yet we can prove that the field cannot grow much more than exponentially, which is in fact enough to our purposes.

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- Beyond the proof of this result, the Ricci hinge Lemma can be applied more generally to estimate the growth of any vector field X along which one can control $\mathcal{L}_X g$, as for instance Killing vector fields.

Ending the proof: the final integration by parts

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- Recall that, letting v an extremal of $\mathcal{L}$ on (M, g), $f = -\frac{m}{2}\ln(4\pi) - 2\ln v$, and $T = i_{\nabla f - X}(\mathrm{Ric} + \mathrm{Hess}(f) - \lambda_S g)$, we have that
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we have that
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- Recalling the expression for f, we hence only have to check that
$$\int_M v^2\left(\mathrm{Ric} - 2\frac{\mathrm{Hess}(v)}{v} + 2\frac{dv\otimes dv}{v^2} - \lambda_S g\right)\left(\frac{\nabla v}{v} - X, \nabla\chi_n\right) d\mathrm{vol} \to 0, \text{ as } n\to\infty.$$
and this can be proved using the properties of χ_n, v and $|X|$.

Thank you!

References

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