Triviality results for quasi-Einstein metrics and Einstein warped products

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Introduction

Problem (Besse)

Determine when one can construct examples of Einstein manifolds which are warped products.

Our results rely on the link beetween Einstein warped product metrics and quasi-Einstein metrics.

Our setting $\rightarrow$ Weighted manifolds.

The main tools are

- Weak Omori–Yau maximum principle for the weighted laplacian.
- Weighted L^p Liouville–type theorems.
- Gradient estimates for solutions of weighted Poisson equations.

$\Rightarrow$ Study of the complete non-compact case.

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Some preliminars

Let $(M, \langle\, ,\, \rangle)$ be a complete m–dimensional Riemannian manifold. Fix an origin $o \in M$ and denote by

- $r(x)$ the distance function from o.
- B_r the geodesic ball of radius r centered at o.
- ∂B_r its boundary.
- $d\mathrm{vol}$ the Riemannian volume density on M.

Given a function $f \in C^\infty(M)$ we can consider the **weighted manifold** $(M, \langle\, ,\, \rangle, e^{-f} d\mathrm{vol})$.
We set

$$\mathrm{vol}_f(B_r(p)) = \int_{B_r(p)} e^{-f} d\mathrm{vol}, \qquad \mathrm{vol}_f(\partial B_r(p)) = \int_{\partial B_r(p)} e^{-f} d\mathrm{vol}_{m-1}.$$

We call f-**laplacian**, Δ_f, the diffusion operator defined on u by

$$\Delta_f u = e^f div\left(e^{-f} \nabla u\right) = \Delta u - \langle \nabla f, \nabla u \rangle$$

which is clearly symmetric on $L^2(M, \langle\, ,\, \rangle, e^{-f} d\mathrm{vol})$.

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A basic principle in Riemannian geometry is that a lower bound on the Ricci curvature implies that the Riemannian measure is bounded above by the measure in a corresponding model space.

A way to expand this principle to the setting of weighted manifolds is to consider the corresponding Ricci tensor, that is the k–**Bakry–Emery Ricci tensor**

$$Ric_f^k = Ric + Hess(f) - \frac{1}{k} df \otimes df, \qquad k > 0.$$

As an application of the Bochner formula for Ric_f^k one can develop

- Myers'–type theorems, (see [Qian, '96], [R., '10]).
- f–laplacian and weighted volume comparisons, (see [Setti '92], [Qian '96], [Wei–Wylie, '09], [Mari–Rigoli–Setti, '10]).

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Quasi-Einstein manifolds

Definition (Case-Shu-Wei '08)

We call a metric k-**quasi-Einstein** if the k-Bakry-Emery Ricci tensor satisfies the equation

$$Ric_f^k = Ric + Hess(f) - \frac{1}{k} df \otimes df = \lambda g_M, \tag{1}$$

for some $\lambda \in \mathbb{R}$.

This equation is especially interesting since

- When $k = \infty$ it is exactly the gradient Ricci soliton equation (important in the study of singularities which arise along the Ricci flow),

  $$Ric_f = Ric + Hess(f) = \lambda g_M$$

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- When f is constant, it gives the Einstein equation and we call the quasi-Einstein metric trivial.
- When $k \in \mathbb{N}$, it corresponds to warped product Einstein metrics.

Note that if $\lambda > 0$, by Qian's version of Myers' Theorem, the quasi-Einstein manifold is compact.

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Link with Einstein warped products

The following characterization holds

Theorem (Kim–Kim '03, Case–Shu–Wei '08)

Let $M^m \times_u F^k$ be an Einstein warped product with Einstein constant λ, warping function $u = e^{-\frac{f}{k}}$ and Einstein fibre F^k. Then the weighted manifold $(M^m, g_M, e^{-f}d\mathrm{vol})$ satisfies the quasi-Einstein equation (1). Furthermore the Einstein constant μ of the fibre satisfies

$$\Delta f - |\nabla f|^2 = k\lambda - k\mu e^{\frac{2}{k}f}. \tag{2}$$

Conversely if the weighted manifold $(M^m, g_M, e^{-f}d\mathrm{vol})$ satisfies (1), then f satisfies (2) for some constant $\mu \in \mathbb{R}$. Consider the warped product $N^{m+k} = M^n \times_u F^k$, with $u = e^{-\frac{f}{k}}$ and Einstein fibre F with ${}^F Ric = \mu g_F$. Then N is Einstein with ${}^N Ric = \lambda g_N$.

Remark

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Examples

In the book by Besse are constructed examples of Einstein warped products with

- $\lambda < 0$ and μ of arbitrary sign,
- $\lambda = 0$ and $\mu \geq 0$.

Moreover, in the latter case, all non-trivial examples have $\mu > 0$, while the trivial quasi-Einstein metrics with $\lambda = 0$ necessarily satisfy $\mu = 0$.

- (Lu–Page–Pope) Other non–trivial examples with $\lambda > 0$, $k > 1$ and $\mu > 0$.

Since, if $k < \infty$ and $\lambda > 0$, M is necessarily compact, the maximum principle applied to

$$\Delta f - |\nabla f|^2 = k\lambda - k\mu e^{\frac{2}{k}f}$$

yields that $\mu > 0$ in this situation.

Weak Omori–Yau maximum principle for Δ_f

We recall that given $\left(M, \langle\, ,\, \rangle, e^{-f} d\mathrm{vol}\right)$ we say that the weak Omori–Yau maximum principle for Δ_f holds if given a C^2 function $u : M \to \mathbb{R}$ satisfying $\sup_M u = u^* < +\infty$, there exists a sequence $\{x_n\} \subset M$ along which

$$(i)\ u(x_n) \geq u^* - \frac{1}{n} \quad \text{and} \quad (ii)\ \Delta_f u(x_n) \leq \frac{1}{n}.$$

Theorem (Pigola–Rigoli–Setti, '05)

Let $\left(M, \langle\, ,\, \rangle, e^{-f} d\mathrm{vol}\right)$ be a geodesically complete weighted manifold satisfying the volume growth condition

$$\frac{r}{\log \mathrm{vol}_f(B_r)} \notin L^1(+\infty). \tag{3}$$

Then, the weak Omori–Yau maximum principle for the f-Laplacian holds on M.

Corollary

Let $\left(M, \langle , \rangle, e^{-f} d\mathrm{vol}\right)$ be a k-quasi-Einstein manifold. Then, the weak Omori–Yau maximum principle for the f-Laplacian holds on M.

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Scalar curvature estimates and some rigidity

Theorem (R. , '10)

Let $(M^m, g_M, e^{-f}\mathrm{dvol})$ be a geodesically complete k–quasi–Einstein manifold, $1 < k < +\infty$, with scalar curvature S and let $S_* = \inf_M S$.

(a) *If $\lambda > 0$, then M is compact and*

$$\frac{m(m-1)}{m+k-1}\lambda < S_* \leq m\lambda. \tag{4}$$

Moreover $S_* \neq m\lambda$ unless M is Einstein.

(b) *If $\lambda = 0$ and $\inf_M f = f_* > -\infty$ then $S_* = 0$. Moreover, either $S > 0$ or $S(x) \equiv 0$. In this latter case, either f is constant (and M is trivial) or M is isometric to the Riemannian product $\mathbb{R} \times \Sigma$ where Σ is a Ricci-flat, totally geodesic hypersurface.*

(c) *If $\lambda < 0$ and $\inf_M f = f_* > -\infty$, then*

$$m\lambda \leq S_* \leq \frac{m(m-1)}{m+k-1}\lambda \tag{5}$$

and $S(x) > m\lambda$ unless M is Einstein.

Improves results in [Case, Shu and Wei, preprint '08].

Triviality of Einstein warped products in case of a compact base

In the book by Besse is faced off the problem of searching new examples of compact Einstein spaces. It is formulated the following

Question (Besse)

Does there exist a compact Einstein warped product with non constant warping function?

D.–S. Kim and Y.–H. Kim gave a negative partial answer to this question.

Theorem (Kim–Kim '03)

Let $N = M \times_u F$ be an Einstein warped product with Einstein fibre F and compact base M. If the scalar curvature of N is non–positive then the warped product is simply a Riemannian product.

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Extension to the case of a non–compact base

The following theorem extends such a conclusion to the case of non–compact bases.

Theorem A (R. '10)

Let $N^{m+k} = M^m \times_u F^k$, $k > 1$, be a complete Einstein warped product with non-positive scalar curvature ${}^N S \leq 0$, warping function $u(x) = e^{-\frac{f(x)}{k}}$ satisfying $\inf_M f = f_* > -\infty$ and complete Einstein fibre F. Then N is simply a Riemannian product if either one of the following further conditions is satisfied:

(a) *f has a local minimum.*

(b) *the base manifold M is complete and non-compact, the warping function satisfies $\int_M |f|^p e^{-\frac{f}{k}} d\mathrm{vol} < +\infty$, for some $1 < p < +\infty$, and $f(x_0) \leq 0$ for some point $x_0 \in M$.*

Note that, in case M is compact, from the point (a) we recover the result of Kim-Kim.

The proof is from the viewpoint of quasi–Einstein metrics and makes an essential use of a weighted L^p–Liouville–type theorem.

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Weighted L^p–Liouville–type theorems

It is a well known result of S.-T. Yau that a non-negative, L^p-subharmonic function, $1 < p < +\infty$, on a complete Riemannian manifold must be constant. This classical Liouville–type theorem has been extended in various directions to both linear and non-linear operators.
Here we recall the following version for the f-Laplacian.

Theorem (Pigola-Rigoli-Setti '05)

Let $\left(M^m, \langle\, , \rangle, e^{-f} d\mathrm{vol}\right)$ be a geodesically complete weighted manifold. Assume that $u \in Lip_{loc}(M)$ satisfy

$$u\Delta_f u \geq 0, \text{ weakly on } (M, \langle\, , \rangle\, e^{-f} d\mathrm{vol}).$$

If, for some $p > 1$,

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Observe that

$$u \in L^p\left(M, e^{-f} d\mathrm{vol}\right) \quad \Rightarrow \quad \frac{1}{\int_{\partial B_r} |u|^p e^{-f} d\mathrm{vol}_{m-1}} \notin L^1(+\infty)$$

Note also that no sign condition is required on u. Moreover, if the locally Lipschitz function u satisfies both $\Delta_f u \geq 0$ and the non-integrability condition then, applying the theorem to $u_+ = \max\{u, 0\}$, gives that either u is constant or $u \leq 0$.

Triviality result for quasi–Einstein manifolds under L^p conditions

Using the scalar curvature estimates for quasi-Einstein manifolds recalled above we can get conditions on the potential f to have triviality for (nonnecessarily compact) k–quasi–Einstein metrics with $k < +\infty$ and $\lambda \leq 0$.

Theorem (R. '10)

Let $(M^m, g_M, e^{-f}d\mathrm{vol})$ be a geodesically complete non-compact k–quasi–Einstein manifold, $1 \leq k < +\infty$. If the quasi-Einstein constant λ is non-positive, f satisfies, for some $1 < p < +\infty$,

$$f \in L^p(M, e^{-\frac{f}{k}}d\mathrm{vol}), \tag{6}$$

and $\inf_M f = f_* > -\infty$, then either $f \equiv const \leq 0$ and M is Einstein or $f > 0$.

Triviality result for quasi–Einstein manifolds under L^p conditions

Using the scalar curvature estimates for quasi-Einstein manifolds recalled above we can get conditions on the potential f to have triviality for (nonnecessarily compact) k–quasi–Einstein metrics with $k < +\infty$ and $\lambda \leq 0$.

Theorem (R. '10)

Let $(M^m, g_M, e^{-f} d\mathrm{vol})$ *be a geodesically complete non-compact k–quasi–Einstein manifold,* $1 \leq k < +\infty$*. If the quasi-Einstein constant* λ *is non-positive, f satisfies, for some* $1 < p < +\infty$,

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and $\inf_M f = f_* > -\infty$*, then either* $f \equiv$ *const* ≤ 0 *and M is Einstein or* $f > 0$.

Proof. Tracing the quasi–Einstein equation and letting $\hat{f} = \frac{1}{k} f$ we have that

$$\Delta_{\hat{f}} f = m\lambda - {}^{M}S.$$

Since $\lambda \leq 0$ and $f_* > -\infty$ we know that

$$m\lambda \leq {}^{M}S_* \leq \frac{m(m-1)}{m+k-1}\lambda.$$

We then obtain that $\Delta_{\hat{f}} f \leq 0$ and by applying the L^p–Liouville result to $f_- = \max\{-f, 0\} \in L^p(M, e^{-\hat{f}} d\mathrm{vol})$ we find that f_- is constant.
Hence if there exist a point $x_0 \in M$ such that $f(x_0) \leq 0$ then $f \equiv f(x_0) \leq 0$.

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On local minima

From the proof, by the strong minimum principle, it follows that if either M is compact or f attains its absolute minimum then f is constant.
Actually the same conclusion holds if we merely assume that f attains a local minimum at some point $x_0 \in M$.
Indeed the following proposition holds.

Proposition (R. '10)

Let $(M, g_M, e^{-f}\mathrm{dvol})$ be a geodesically complete non-compact k-quasi-Einstein manifold, $1 < k < +\infty$. If the quasi–Einstein constant λ is non positive and f satisfies $f_* > -\infty$, then any local minimum of f is actually an absolute minimum.

Proof. Assume that f attains a local minimum $x_0 \in M$. Evaluating $\Delta_{\hat{f}} f = m\lambda - {}^M S$ at x_0, we get ${}^M S(x_0) \leq m\lambda$.
Hence, since $\lambda \leq 0$, by the rigidity part of our scalar curvature estimates, M is Einstein and ${}^M S$ is identically $m\lambda$.
Thus the quasi-Einstein equation reads

$$Hess(f) = \frac{1}{k} df \otimes df.$$

In particular $Hess(f)$ is positive semi-definite on M and this implies the thesis.

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Proof of Theorem A

The proof of the triviality theorem for Einstein warped product now easily follows from the link between these and quasi–Einstein metrics.

Indeed in case (a) the triviality of f and hence of the warping function $u = e^{-\frac{f}{k}}$ follows immediately from the considerations above.

On the other hand, in case (b), o since N is Einstein $\lambda = \frac{1}{m+k}\,{}^{N}S \leq 0$ and we can apply the triviality result we have proven for quasi–Einstein metric.

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Other Triviality results

In [Case,'09] triviality of quasi-Einstein metrics is dealt by considering only the equation

$$\Delta f - |\nabla f|^2 = k\lambda - k\mu e^{\frac{2}{k}f}.$$

Theorem (Case '09)

Let $N^{m+k} = M^m \times_u F^k$ be a complete Einstein warped product with warping function $u(x) = e^{-\frac{f(x)}{k}}$, scalar curvature ${}^N S \geq 0$ and complete Einstein fibre F. Then N is simply a Riemannian product provided the base manifold M is complete and the scalar curvature of F satisfies ${}^F S \leq 0$.

In the following theorem we obtain the same conclusion in case ${}^F S \geq 0$ up to assume an integrability condition on the warping function u.

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Observe that non-trivial examples with ${}^N S \leq 0$ and ${}^F S \geq 0$ are constructed in the book by Besse. Thus the integrabilty assumption is necessary.

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Proof. Just observe that computing the f-laplacian of $u = e^{-\frac{f}{k}}$ and using the equation that relates the Einstein constant of the product and that of the fibre, one obtains the following equation

$$\Delta_f u = \mu u^{-1} - \lambda u + \frac{u}{k^2}|\nabla f|^2.$$

Thus, in our assumptions, we obtain that $\Delta_f u \geq 0$.
Since $0 < u \in L^p(M, e^{-f} d\mathrm{vol})$, by the general weighted L^p–Liouville theorem stated above, we obtain the constancy of u and up to a rescaling of the metric of F we can suppose $u = 1$.
Now, since the Riemannian product $M \times F$ is Einstein, both M and F are Einstein manifolds with the same Einstein constant. In particular, ${}^M S$ and ${}^F S$ have the same sign. By our assumption on the signs of ${}^N S$ and ${}^F S$ we thus obtain that both M and F are Ricci flat.
Finally, since u (and thus f) is constant, from the integrability condition we obtain that $vol(M) < +\infty$. Thus, by a result of Calabi-Yau we obtain that M must be compact.

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Applying again the weak maximum principle for the f–laplacian to the equation for the warping function

$$\Delta_f u = \mu u^{-1} - \lambda u + \frac{u}{k^2}|\nabla f|^2,$$

one can prove also the following non–existence result

Theorem (R.'10)

There is no complete Einstein warped product $N = M^m \times_u F^k$ with warping function $u = e^{-\frac{f}{k}} \in L^\infty(M)$, scalar curvature ${}^N S < 0$ and Einstein fibre F with ${}^F S \geq 0$.

Triviality by gradient estimates

The proof of Case's triviality theorem is a consequence of the following gradient estimate for solutions of weighted Poisson equations.

Theorem (Case '09)

Let $(M^m, g_M, e^{-f}\mathrm{dvol})$ *be such that* $Ric_f^k \geq 0$, $k < \infty$, *and*

$$\Delta_f f = \phi(f),$$

where $\phi : \mathbb{R} \to \mathbb{R}$ *is a function such that*

$$\phi'(t) + \frac{2}{m}\phi(t) \geq 0$$

for all $t \in \mathbb{R}$. *Then for all* $q \in M$, $T > 0$ *such that* $B(q, T)$ *is geodesically connected in* M *and the closure* $\overline{B(q, T)}$ *is compact,*

$$|\nabla f|^2(q) \leq \frac{mk}{m+k}\frac{2(m+k+6)}{T^2}.$$

We recently obtained a similar estimate even in case $\lambda < 0$.

Theorem (Mastrolia, R. '10)

Let $(M^m, g_M, e^{-f} d\mathrm{vol})$ *be a weighted manifold. Suppose that, for some* $k < +\infty$, $Z > 0$,

$$Ric_f^k \geq \lambda = -(m+k-1)Z^2$$

and that

$$\Delta_f f = \psi(f),$$

where $\psi : \mathbb{R} \to \mathbb{R}$ *satisfy*

$$\psi'(t) + \frac{2}{m}\psi(t) + \lambda \geq 0$$

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$$|\nabla f|^2(q) \leq \frac{mk}{m+k}\left[\frac{2(m+k+6)}{T^2} + \frac{4\sqrt{3}}{9}\frac{(m+k-1)Z}{T}\right].$$

Remark

Note that in case $Ric_f^k \geq 0$ *we recover Case's result by letting* $Z \to 0^+$.

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Remark

Note that in case $Ric_f^k \geq 0$ *we recover Case's result by letting* $Z \to 0^+$.

Proof.(Sketch) By the Bochner formula for Ric_f^k and the hypothesis on ψ we get

$$\Delta_f |\nabla f|^2 \geq 2\left(\frac{1}{m}+\frac{1}{k}\right)|\nabla f|^4. \tag{7}$$

Let now $\rho(x) = \mathrm{dist}(q, x)$. Using a trick by Calabi we can suppose that ρ is smooth. Consider the function on $B(q, T)$ defined by

$$F(x) = \left[T^2 - \rho^2(x)\right]^2 |\nabla f|^2.$$

If $|\nabla f| \equiv 0$ we have nothing to prove. So suppose that $|\nabla f| \not\equiv 0$. Since $F \geq 0$ and $F|_{\partial B(q,T)} \equiv 0$, there exists a point $x_0 \in B(q, T)$ such that $F(x_0) = \max_{B(q,T)} F(x) > 0$. At x_0 we then have

$$\frac{\nabla F}{F}(x_0) = 0, \quad \frac{\Delta_f F}{F}(x_0) \leq 0.$$

Using (7), the definition of F and the f–laplacian comparison, some computations give that at x_0

$$2\left(\frac{1}{m}+\frac{1}{k}\right)\left[T^2-\rho^2\right]^2|\nabla f|^2 \leq 4(m+k-6)T^2 + \frac{8\sqrt{3}}{9}(m+k-1)ZT^2,$$

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As a corollary of this gradient estimate we immediately get to the following Liouville-type theorem for Einstein warped products.
The proof is straightforward once we choose $\psi(t) = k\lambda - k\mu e^{\frac{2}{k}t}$ and imposing the validity of the differential inequality of the theorem.

Theorem (Mastrolia, R. '10)

Let $N = M^m \times_u F^k$ a complete Einstein warped product with warping function $u = e^{-f/k}$, scalar curvature ${}^N S = (m+k)\lambda < 0$ and complete Einstein fibre F^k with scalar curvature ${}^F S = k\mu < 0$. Suppose that

$$f \geq \frac{k}{2} \log\left(\frac{\lambda}{2\mu}\frac{m+2k}{m+k}\right) \qquad \text{for all } x \in M.$$

Then N is simply a Riemannian product (up to a rescaling of the metric on F).

Thank you!

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